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A. Dyson

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FLOW PROPERTIES OF MINERAL OILS IN ELASTOHYDRODYNAMIC LUBRICATION

By A. DYSON

'Shell' Research Ltd, Thornton Research Centre, P.O. Box 1, Chester

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CONTENTS

From experimental results obtained by Crook (1963) and by Smith (1960) from the field of elastohydrodynamic lubrication, it must be concluded that in certain circumstances mineral oils exhibit non-Newtonian behaviour, i.e. the shear stress is no longer proportional to the shear rate, at constant temperature. In this paper, we seek to describe the relations between shear stress

and shear rate, which have been reported for mineral oils under various experimental conditions, and to relate them to properties of mineral oils reported from other fields. We start with the supposition that the non-Newtonian behaviour is caused by the viscoelastic properties of the oil.

From the theory given by Oldroyd (1958) of the behaviour of a viscoelastic substance in continuous shear at high shear rates, it is concluded that, under certain simplifying conditions, there should be a simple relation between the stress system in continuous shear at a shear rate D, and the components of the complex shear modulus G^* in oscillatory shear at an angular frequency ω, with $\omega = KD$. Here K is an unknown shift constant.

This conclusion is confirmed within a certain range of conditions by an examination of experimental results, particularly those of Russell (1946), obtained with substances showing viscoelastic behaviour under conditions more easily reproducible under controlled laboratory conditions than those required for mineral oils. We then consider the results for mineral oils obtained by Crook (1963) and by Smith (1960) under conditions of elastohydrodynamic lubrication. These are fitted to the results obtained in oscillatory shear by Barlow & Lamb (1959) and satisfactory agreement is obtained.

Whereas for other viscoelastic materials $K \simeq 1$, apparently independent of pressure, we find that for mineral oils K decreases with increasing pressure and increases with increasing temperature. Extrapolation to atmospheric pressure gives $K \simeq 10^3$, which is consistent with the results obtained by Porter & Johnson (1959), who used a rotational viscometer.

A tentative interpretation of the variation of K in terms of that of the Oldroyd parameter μ_0 is given. Certain features of the experimental evidence so far available are unsatisfactory, and more work is needed.

NOTATION

b	half-width of Hertzian contact (cm)
C^*	complex shear compliance $= 1/G^*$
C'	real part of complex shear compliance
C''	imaginary part of complex shear compliance
D	shear rate (s^{-1}) in simple shear
E_0	$\lim_{\omega \to 0} (1/C')$
$f(\tau)$	function entering relation between shear stress and shear rate; $D = (\tau/\eta_0) f(\tau)$
f_b	coefficient of boundary friction
G_∞	shear modulus of Maxwell body at high frequencies
G^*	complex shear modulus
G'	real part of complex shear modulus
G''	imaginary part of complex shear modulus
h	oil film thickness (cm)
j	operator denoting imaginary component in oscillatory phenomena
k	thermal conductivity of oil (ergs $\mathrm{s}^{-1}\,\mathrm{cm}^{-1}\,\mathrm{degC}^{-1}$)
K	shift factor entering relation between behaviour in oscillatory and in continuous shear, $\omega = KD$
p	isotropic pressure ($\mathrm{dyn/cm}^2$)
P_{ik}	total stress tensor ($\mathrm{dyn/cm}^2$)
P'_{ik}	stress tensor after superposition of isotropic pressure p
t	time (sec)
U	velocity (cm/s)
U_1, U_2	velocities (cm/s) of surfaces 1 and 2 relative to the contact between them

u_1, u_2, u_3	velocities (cm/s) of particle of lubricant parallel to x_1, x_2, x_3
x_1, x_2, x_3	co-ordinates in direction of motion, in direction of velocity gradient, and in transverse direction, respectively
z_1	$\eta_0 D/G_\infty$
z_2	$\eta_0 \omega/G_\infty$
α	coefficient in exponential viscosity-temperature equation (P) ($\eta = \alpha e^{-\beta\theta}$)
β	temperature coefficient of viscosity (deg C^{-1}) ($\eta = \alpha e^{-\beta\theta}$)
γ	pressure coefficient of viscosity (dyn^{-1} cm^2) ($\eta = (\eta)_0 e^{\gamma p}$)
η	viscosity (P)
$(\eta)_0$	viscosity (P) at atmospheric pressure
η_0	viscosity (P) at low shear rate
η'	effective viscosity, (P), such as would give the observed shear stress if applicable to all the lubricant present in the film in the Hertzian contact region
η^*	$= G^*/j\omega$, complex viscosity in oscillatory shear
$\bar{\eta}$	resultant viscosity, (P), obtained by averaging over a pressure distribution
θ	temperature (°C)
λ_1	relaxation time (sec)
λ_2	retardation time (sec)
$\mu_0, \mu_1, \mu_2, \nu_1, \nu_2$	constants in Oldroyd equation, all with dimensions of time (sec)
τ	shear stress (dyn/cm^2)
ω	angular frequency (rad/s)

1. Introduction

The formation of the lubricating oil film necessary for the correct functioning of gears, cams and tappets, and other heavily loaded components depends on a mechanism known as elastohydrodynamic lubrication. The thickness of the lubricant film may be predicted approximately for minerals oils by theories which treat oil as a Newtonian fluid, i.e. a fluid having a viscosity independent of the shear rate, and in which temperature variations across the film are ignored.

These temperature variations across the film are important, however, in the consideration of the coefficient of sliding friction and at high speeds they dominate the variation of the coefficient of friction with sliding speed. The friction is then determined mainly by the thermal conductivity and by the viscosity-temperature characteristics of the oil.

At lower sliding speeds, however, there is a region in which the variation of friction with sliding speed cannot be explained on the basis of temperatures developed within a film of Newtonian lubricant. It is therefore necessary to abandon the original assumption of Newtonian behaviour, and to admit a non-linear relation between shear stress and shear rate. This conclusion is evident from the results of Crook (1963) and of Smith (1960).

The acceptance of this conclusion raises a number of points concerning the form of the relation between shear stress and shear rate. How does this relation vary with conditions such as temperature and pressure? Is the form of the relation under any given conditions to be regarded as an arbitrary fact of nature, or are there general principles which govern the phenomenon as a whole? Can it be related to properties exhibited by an oil in other fields?

It is with these questions that this paper is concerned. We first discuss the difficulties encountered in the interpretation of the results from studies in elastohydrodynamic lubrication, and find that these difficulties are even more formidable when an attempt is made to reproduce this behaviour in laboratory viscometers under controlled conditions.

We therefore enquire whether the phenomenon is related to other properties which may perhaps be more easily measured without difficulties in interpretation. We start with the supposition that the non-Newtonian behaviour exhibited by mineral oils in elastohydrodynamic lubrication is caused by viscoelasticity. With certain simplifying assumptions, we find that there is a theoretical relation between the behaviour of a viscoelastic substance in oscillatory and in continuous shear, and we investigate the applicability of this relation to the results obtained by Russell (1946), with substances which show viscoelastic properties under conditions more easily obtainable in the laboratory than those required for mineral oils.

We then compare the relations between shear stress and shear rate for mineral oils obtained in elastohydrodynamic lubrication by Crook (1963) and by Smith (1960) with the results in oscillatory shear obtained by Barlow & Lamb (1959). Unfortunately these authors used different minerals oils in their investigations. There is an obvious need for experimental work covering the behaviour of the same fluids in both continuous and oscillatory shear, and it is hoped to undertake this eventually. It appears unlikely that any results will be available in the immediate future, however, and there are some grounds for supposing that the viscoelastic behaviour of different mineral oils, of the type used in these investigations, does not vary widely. It therefore seems worth while to present those comparisons which can be made at this stage.

We are not concerned here with non-Newtonian behaviour of the type encountered in a mineral oil at low temperatures as a result of phase separation, wax formation, etc., or of the type produced by the solution of high-molecular weight long-chain polymers in the oil. Such behaviour is readily accessible to observation, and is well understood in principle. We are concerned with phenomena occurring in the oil itself treated as a continuous medium. The observation of such phenomena is difficult and the interpretation is subject to controversy.

2. Direct experimental evidence of non-Newtonian behaviour of mineral oils

Direct experimental evidence of the relation between shear stress and shear rate is very difficult to obtain. That obtained from the field of elastohydrodynamic lubrication suffers from a number of disadvantages, in that the oil is subject to a distribution of pressure which is known only in approximation. There is also the fundamental difficulty that the heat generated by shear in the oil film will increase the oil film temperatures and therefore reduce the apparent viscosity at high sliding speeds. For a given shear stress and shear rate, the temperatures generated in the film are proportional to the square of the film thickness (Archard 1958/9). Now the oil films in elastohydrodynamic lubrication have a thickness of the order of 10^{-4} cm or less, whereas the thinnest films used in laboratory viscometers are of the order of 10^{-3} cm. If attempts are made to use appreciably thinner oil films at high shear rates, difficulties arise which are associated with thermal distortion of the gap, elastic deformations of the walls under varying pressure, etc. Furthermore, it is difficult to operate a rotational viscosmeter under a hydrostatic pressure of the order of hundreds of thousands of pounds per square inch. It is therefore very difficult to reproduce the phenomenon under controlled laboratory conditions in which pressure and shear rate are constant and known. No information seems to be available under the conditions of pressure and temperature which are of interest for the study of elastohydrodynamic lubrication.

Some results for mineral oils under pressure have been obtained by Norton, Mott & Muenger (1941) using capillary tube viscometers, and have been analysed by Hahn, Eyring, Higuchi & Ree (1958). These results indicate departures from Newtonian behaviour at shear stresses of the order of 10^5 to 10^6 dyn/cm^2 at pressures up to about 2×10^9 dyn/cm^2 (actually 2040 atm.) These conclusions probably exaggerate the departure from Newtonian behaviour, since no corrections appear to have been made for the effects of temperature rise in the oil due to heating, or for the variation of viscosity with pressure along the length of the capillary tube.

In their reply to the discussion on their paper, Hahn *et al.* showed that such corrections considerably reduced the apparent departure from Newtonian behaviour exhibited by castor oil. Since the temperature correction was only approximate in nature, their results seem to be consistent with Newtonian behaviour of castor oil under their experimental conditions. A similar result would probably follow from the application of these corrections to the data for the mineral oil.

We conclude that there is no direct experimental evidence from fields other than that of elastohydrodynamic lubrication, of non-Newtonian behaviour of mineral oils under the conditions pertaining in this service. We therefore enquire whether we can relate this phenomenon to properties of the fluids exhibited in other fields, for which data may be available, and this raises the question of the physical processes underlying this non-Newtonian behaviour.

3. Viscoelasticity

(*a*) *The Maxwell model*

We suppose that the physical basis underlying the non-Newtonian behaviour of a mineral oil is viscoelasticity, that is, the appearance of an additional mode of deformation of the substance, in addition to the more usual mode of deformation as a Newtonian liquid. The true viscosity of the liquid is in fact constant (at constant temperature), but the apparent viscosity is reduced as a result of this additional deformation.

The simplest model of a viscoelastic material is the Maxwell substance. It is assumed that the solid-like mode of deformation is controlled by an elastic shear modulus G_∞, and that this deformation is added to the deformation as a Newtonian liquid, controlled by a coefficient of viscosity η_0. The equation representing the deformation of such a substance under an applied shear stress τ is then

$$D = \frac{\tau}{\eta_0} + \frac{1}{G_\infty}\frac{\mathrm{d}\tau}{\mathrm{d}t}, \tag{1}$$

where D is the shear rate and t is time.

If this equation is applied in a system of reference axes which maintains its orientation relative to the planes of shear, it is found that viscoelastic effects become important when the time of transit of the oil through the contact is of the same order as the relaxation time, η_0/G_∞ (Milne 1957; Crouch & Cameron 1960).

This condition is

$$\frac{2b}{U} \simeq \frac{\eta_0}{G_\infty},$$

where b is the half-width of the Hertzian contact and $\bar{U}$ is the mean velocity of the surfaces relative to the contact between them.

If $\bar{U} = 10^2$ cm/s, $b = 10^{-2}$ cm, $G = 10^{10}$ dyn/cm^2 (Barlow & Lamb 1959), this condition is not met until $\eta_0 \simeq 10^6$ P. This is obviously far removed from practice, as the shear stress for a sliding speed of 10 cm/s and a film thickness of 10^{-4} cm would be 10^{11} dyn/cm^2, or approximately 1 500 000 lb./in.^{2}. It therefore appears that relaxation effects, formulated in this way, cannot explain the appearance of non-Newtonian behaviour in elastohydrodynamic lubrication.

However, many authors, e.g. Fromm (1948) and Oldroyd (1950) have pointed out that equation (1) above cannot be applied in a system of axes which maintains its orientation relative to the planes of shear. The essential physical reason is that a simple shear deformation includes a component of rotation of the elements of the substance as a whole. This rotation should not influence the stresses on the element, and it is necessary to apply equation (1) in a system of axes which follow the rotation of an element of the fluid. The angular velocity of rotation of these axes is $\frac{1}{2}D$ rad/s, where D s^{-1} is the rate of simple shear, or the velocity gradient, referred to the usual stationary axes. A simple and accessible discussion of this point is given by Tanner (1960) and an equivalent argument is given in appendix A, in which the notation established in section (*b*) below is used.

The rotation of the axes introduces fresh terms in $d\tau/dt$ and therefore additional elastic deformations. When the stresses required to produce the assumed velocity gradient are calculated in the rotating system and then referred back to the original axes, three noteworthy effects are found. First, the shear stress increases with increasing shear rate much less rapidly than would be expected from a direct application of equation (1) in the original axes, i.e. the viscosity decreases with increasing shear rate much more rapidly than would be expected from the straightforward calculation. Secondly, normal stress differences appear (Weissenberg effect), and thirdly, there is a very close relation between the behaviour of the substance in continuous shear and in oscillatory shear. Experiments in oscillatory shear may be carried out in the laboratory under controlled conditions without difficulties due to heating of the liquid, and the behaviour of mineral oils in oscillatory shear has already been investigated by Barlow & Lamb (1959).

Therefore, if we find we can use the Maxwell model to represent the behaviour of a mineral oil, we can infer its properties in continuous shear, as exhibited in the field of elastohydrodynamic lubrication, from the known properties in oscillatory shear. We shall find that we must modify the Maxwell model before we can do this, and for this discussion we must first establish a system of notation.

(*b*) *Notation*

The principal difficulty is to describe the stress systems arising from the deformation of viscoelastic materials. These stress systems are more complicated than those arising from

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the deformation of Newtonian liquids, and it is convenient to use tensor notation. Unfortunately the conventions used by different authors vary widely, but perhaps the most common is that of figure 1, in which the direction of motion is denoted by 1, the direction normal to the shear planes, i.e. that of the velocity gradient, by 2, and the transverse direction by 3. The force on unit area of a plane normal to the 1 axis is resolved into three components P_{11}, P_{12} and P_{13} parallel to the 1, 2, 3 directions respectively. The forces on planes normal to the 2, 3 directions are resolved in a similar manner. If inertia and gravity are ignored application of the angular momentum equation to an elementary cube of the fluid gives

$$P_{12} = P_{21}; \quad P_{23} = P_{32}; \quad P_{31} = P_{13}.$$

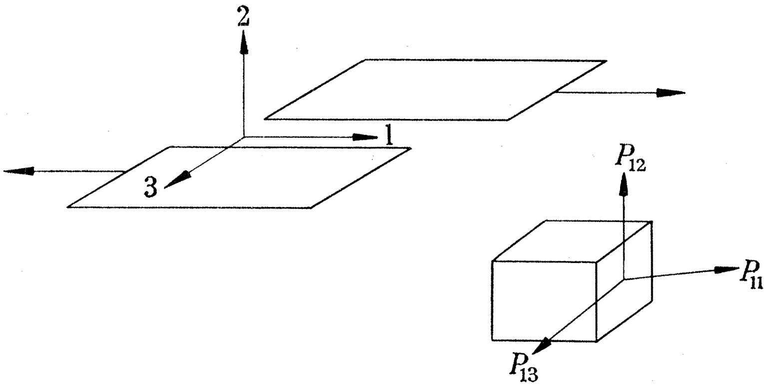

FIGURE 1. Stress system for simple shear.

P_{12} is the shear stress, which we also denote by τ when we do not wish to consider the normal stress differences. The set of nine stress components, six of which are independent, is known as the total stress tensor P_{ik}, where i and k range over the values 1, 2 and 3. The theory is much simpler if the liquid is treated as isotropic and incompressible, and an arbitrary isotropic pressure p may then be superimposed on any stress system without effect on the deformations. The resulting stress tensor is P'_{ik}, where

$$P_{ik} = P'_{ik} - p\delta_{ik},$$

where $\delta_{ik} = 1$ if $i = k$, $\delta_{ik} = 0$ if $i \neq k$.

The sign convention is such that an outward pressure by the liquid on a boundary gives a negative normal component of the stress tensors.

We are now in a position to discuss the detailed results obtained with the Maxwell model, and some well-known objections to the use of this model to represent the behaviour of visco-elastic liquids in continuous shear.

(c) *Objections to the use of the Maxwell model*

The behaviour of a Maxwell substance in continuous simple shear at a shear rate D is given by the following stress system (Fromm 1948; Oldroyd 1958; Tanner 1960)

$$\left.\begin{aligned} \frac{P'_{12}}{G_\infty} &= \frac{z_1}{1+z_1^2}, \\ \frac{P'_{11}}{G_\infty} &= -\frac{P'_{22}}{G_\infty} = \frac{z_1^2}{1+z_1^2}, \\ P'_{33} &= 0, \end{aligned}\right\} \tag{2}$$

where $z_1 = \eta_0 D/G_\infty$.

If the Maxwell body is subject to oscillatory shear at a radian frequency ω, the apparent complex shear modulus G^*, the instantaneous ratio between shear stress and shear strain, may be written

$$\left.\begin{aligned} G^* &= G' + \mathrm{j}G'', \\ \text{where}\quad \frac{G''}{G_\infty} &= \frac{z_2}{1+z_2^2}, \\ \frac{G'}{G_\infty} &= \frac{z_2^2}{1+z_2^2} \\ \text{and}\quad z_2 &= \eta_0\omega/G_\infty. \end{aligned}\right\} \tag{3}$$

Equations (2) and (3) are formally identical with

P'_{12} equivalent to G'', $\tfrac{1}{2}(P_{11}-P_{22}) = P'_{11}$ equivalent to G', D equivalent to ω.

Since the effective dynamic viscosity η' is defined by

$$\eta' = G''/\omega$$

(Barlow & Lamb 1959), the variation of effective viscosity with angular frequency in oscillatory shear is identical with its variation with shear rate in continuous shear, with ω simply replaced by D.

Most real substances show a variation of G' and G'' with frequency which is less rapid than that given above for a Maxwell body, and it is usual to explain this as a result of the parallel action of a number of individual Maxwell elements, with a distribution of relaxation times. This explanation is supported by the agreement of the two distributions of relaxation times, calculated independently from the variation with frequency of G' and G'' (Barlow & Lamb 1959). The above equivalence of the behaviour of the substance in continuous and oscillatory shear will be valid separately for each component of the distribution, and therefore for the real substance as a whole.

The simplicity of this theoretical conclusion is attractive, but there are three major and well-known objections to its acceptance:

(i) As the initial viscosity η_0 or the rate of shear D is increased, the shear stress predicted by the above theory reaches a maximum and then decreases. This behaviour seems improbable on intuitive grounds, and it may lead to a peculiar instability of the flow pattern (Fromm 1948; Smith 1960).

(ii) It has been demonstrated that real viscoelastic fluids, e.g. polymer or soap solutions, give a normal stress distribution which is isotropic in a plane normal to the direction of motion, i.e. $P'_{22} = P'_{33}$ (Russell 1946; Roberts 1953; Jobling & Roberts 1958). A simple Maxwell element, or a distribution of such elements, cannot reproduce this behaviour, except in the trivial case where there are no stress differences. This is evident from equations (2).

(iii) For shear stresses of the same order as the elastic modulus G_∞, there will be large elastic strains, whereas the elasticity theory used in the derivation of equations (2) is essentially restricted to small strains.

We shall now discuss how, and to what extent, these objections may be avoided. The theoretical maximum shear stress mentioned in objection (i) is of the same order as G'_∞

and objection (iii) then applies also. This means that the theory may be valid only up to a point somewhat below the shear stress maximum. We shall see later that experiment indicates that this supposition is correct.

(*d*) *The Oldroyd model*

Objections (i) and (ii) above are met by a more general theory of Oldroyd (1958), who considers the class of substances whose behaviour in small-strain oscillatory shear is characterized by a retardation time $\lambda_2 (< \lambda_1)$ in addition to the relaxation time $\lambda_1 (= \eta_0/G_\infty)$ characterizing a Maxwell body. The equation expressing this behaviour is

$$P'_{ik}+\lambda_1 \frac{\mathrm{d}P'_{ik}}{\mathrm{d}t} = 2\eta_0 \left(E_{ik}+\lambda_2 \frac{\mathrm{d}E_{ik}}{\mathrm{d}t}\right), \tag{4}$$

where E_{ik} is the rate-of-strain tensor, defined by

$$E_{ik} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_i}+\frac{\partial u_i}{\partial x_k}\right)$$

and u_i is the component of the velocity of the fluid parallel to the direction x_i. The differential coefficients are unambiguous at low shear rates. We shall regard the retardation time λ_2 as introduced for the sake of generality, and we shall be able to discard it eventually.

Oldroyd considers all the possible equations of state describing the behaviour of such a substance in continuous shear, this description not being limited to low shear rates as is equation (4).

The simplest general equation is very complicated and involves five additional parameters, in addition to η_0, λ_1 and λ_2:

$$\mu_0; \quad \mu_1; \quad \mu_2; \quad \nu_1; \quad \nu_2.$$

There is then no necessary connexion between the behaviour in continuous and in oscillatory shear, since these five additional parameters intervene in continuous shear but not in oscillatory shear.

It is possible to impose some restrictions on the choice of these five parameters, however. Oldroyd shows that the experimental results of Roberts,

$$P'_{22} = P'_{33},$$

require that

$$\mu_1 = \lambda_1; \quad \mu_2 = \lambda_2.$$

There is still no necessary relation between the behaviour in oscillatory and that in continuous shear; in particular, the remaining disposable parameters μ_0, ν_1, and ν_2 may be chosen so that the maximum in the shear stress is suppressed.

So far, we have confined ourselves to a review of published work which is generally accepted, although we have considered it with special reference to the conditions pertaining in elastohydrodynamic lubrication. We now enter a rather more controversial field, in which we hope to make some original contribution.

We start by considering another limitation which may be imposed on the choice of the Oldroyd parameters. Russell (1946) reports that the viscoelastic substances which he investigated gave stress components normal to the direction of motion which were in equilibrium with those in the unsheared liquid, ie. that for which $D = 0$. The remaining normal stress effect is then equivalent to a tension along the streamlines. This result was given as an example by Roberts (1953) and by Jobling & Roberts (1958) but was not stated explicitly as a general conclusion.

It may be shown that an Oldroyd fluid with this property would be subject to the restrictions

$$\nu_1 = 0 = \nu_2$$

and together with the previous restrictions on the choice of μ_1 and μ_2, this condition results in the following expressions for the stress components in simple laminar shear:

$$\left.\begin{aligned} P'_{11} &= \frac{2\eta_0 D^2(\lambda_1 - \lambda_2)}{1+\mu_0\lambda_1 D^2}, \\ P'_{12} &= \frac{\eta_0 D(1+\mu_0\lambda_2 D^2)}{1+\mu_0\lambda_1 D^2}, \\ P'_{12} &= 0 = P'_{33}. \end{aligned}\right\} \tag{5}$$

The behaviour of the general Oldroyd body in small-amplitude oscillatory shear is described by the relations

$$\left.\begin{aligned} G' &= \frac{\eta_0\omega^2(\lambda_1 - \lambda_2)}{1+\lambda_1^2\omega^2}, \\ G'' &= \frac{\eta_0\omega(1+\lambda_1\lambda_2\omega^2)}{1+\lambda_1^2\omega^2}. \end{aligned}\right\} \tag{6}$$

From a comparison of equations (5) and (6), we observe that the shear stress P'_{12} is $1/K$ of the value of G'' at an angular frequency $\omega = KD$, where $K^2 = \mu_0/\lambda_1$. In other words, the variation of apparent viscosity with shear rate D in continuous shear would be the same as the variation of dynamic viscosity (G''/ω) with angular frequency ω in oscillatory shear, with ω replaced by KD.

Furthermore, one-half of the normal stress difference, i.e. $\frac{1}{2}(P_{11} - P_{22})$ in the usual notation, should be $1/K^2$ of the value of G', also at a frequency $\omega = KD$.

The above relations will still hold for a real viscoelastic substance composed of a number of Oldroyd elements with distributions of the relevant parameters, provided the ratio μ_0/λ_1 is the same for each element. Whether this ratio is indeed constant from element to element, and its value under any given conditions, must be determined by experiment. We now investigate this point by considering the results published by a number of workers, but we must first note that an Oldroyd body, subject to all the limitations we have imposed on it, still shows a maximum in the shear stress, i.e. objection (i) above remains. As we have seen, it may be possible to avoid this by the operation of objection (iii) concerning the failure of the elastic theory at the large elastic strains which would be present at the shear stress maximum.

It should also be noted that the parameter λ_2 does not enter into these relations. For most of the fluids of interest in the study of elastohydrodynamic lubrication, G'' tends to zero as ω tends to infinity, and we may therefore put $\lambda_2 = 0$, without loss of generality.

(*e*) *Experimental comparison of the behaviour of viscoelastic substances in oscillatory and in continuous shear*

We now consider published work on this subject, in an effort to determine how closely we can rely on the relations suggested above between the behaviour of real viscoelastic substances in continuous and in oscillatory shear. We must consider substances which exhibit viscoelastic behaviour under conditions more easily realized in the laboratory than those required for mineral oils.

This question has been studied by many experimenters, mostly by a comparison between the variation of apparent viscosity with angular frequency in oscillatory shear and its variation with shear rate in continuous shear. Onogi, Hamana & Hirai (1958) found that these variations were approximately identical, with ω replaced by KD, where K was approximately equal to unity, for concentrated solutions of polyvinylalcohol. They covered a range of values of the viscosity ratio η'/η_0, of the effective viscosity to that at low shear rates or low frequencies, from unity to 0·2. Horio, Onogi & Ogiwara (1961) also found K to be approximately equal to unity for molten polyethylene, down to $\eta'/\eta_0 \simeq 0{\cdot}5$. Onogi, Fujii, Kato & Ogihara (1964) extended this range down to $\eta'/\eta_0 \simeq 0{\cdot}2$, and also gave a useful review. De Witt, Markowitz, Fadden & Zapas (1955) covered a range of η'/η_0 down to approximately 0·05 for solutions of polyisobutylene in decalin. They found good agreement between the viscosities in continuous and in oscillatory shear down to moderate values of the viscosity ratio, approximately 0·2, with $K \simeq 0{\cdot}7$, but at lower values of the viscosity ratio (higher frequency or shear rate), the viscosities in continuous shear became significantly greater than those in oscillatory shear. Similar results were obtained by Philippoff (1954), who extended the range of viscosity ratio down to approximately 10^{-3} for systems of similar constitution. He found that a value of K of approximately 0·4 gave good agreement between the viscosities in continuous and in oscillatory shear down to a viscosity ratio of approximately 0·2, but again, the viscosities in continuous shear became higher than those in oscillatory shear as the frequency and shear rate were increased.

The investigations hitherto described have been concerned only with the comparison of effective viscosities. Comparisons of the normal stress difference in continuous shear with the real part of the complex shear modulus in oscillatory shear have been made by Markovitz & Williamson (1957) and by Markovitz (1957), again for solutions of polyisobutylene. They found good agreement with $K \simeq 0{\cdot}7$ up to moderate stresses, but as the frequency and shear rate were further increased, the normal stress differences in continuous shear became greater than the modulus component in oscillatory shear.

So far as the author is aware, the experimental work that is most suitable for our present purpose is that of Russell (1946), who measured values of η'/η_0 down to approximately 10^{-3}, and furthermore included data for the normal stress comparison. Russell did not make a direct comparison of his results in oscillatory and in continuous shear, and in fact gave the oscillatory shear results in terms of the real and imaginary parts of the complex shear compliance,

$$C^* = C' - jC'' = 1/G^*.$$

It therefore seemed worth while to recalculate his results in terms of the components of the modulus, and to compare these with the shear and normal stresses in continuous shear. Unfortunately he does not give many details of the materials he used, but they may have been solutions of materials of high molecular weight in solvents of low molecular weight, and in this case the results may have been influenced by the viscosity of the solvent. Russell finds that the shear stress in continuous shear at high shear rates depends to a large extent on the limiting value $(C''/C')_\infty$ at high frequencies. From his results, we may select three materials, designated by the symbols AFGO/L, AFGO/H and SLW. 10, which have similar values of $(C''/C')_\infty$ and for which the necessary results in continuous and in oscillatory shear

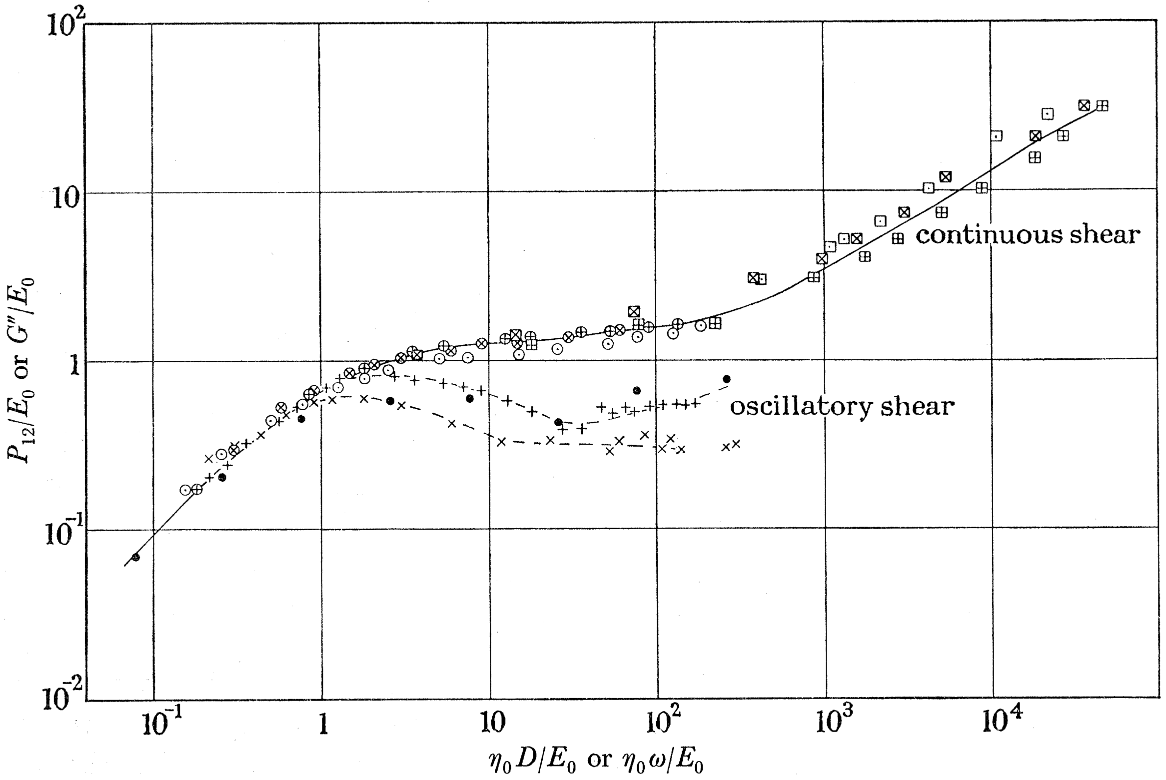

FIGURE 2. Comparison of shear stress in continuous shear with imaginary modulus in oscillatory shear (results of Russell).

		AFGO/H	AFGO/L	SLW.10
oscillatory shear, G''/E_0 against $\eta_0\omega/E_0$		×	●	+
continuous shear, P_{12}/E_0 against $\eta_0 D/E_0$	disk experiments	⊗	⊙	⊕
	capillary experiments	⊠	⊡	⊞

are given over a sufficiently wide range. To normalize his results, Russell used a modulus E_0, defined by

$$E_0 = \lim_{\omega\to 0}\left(\frac{1}{C'}\right) = \lim_{\omega\to 0}\left(\frac{G'^2+G''^2}{G'}\right) \simeq \lim_{\omega\to 0}\left(G'+\frac{\eta_0^2\omega^2}{G'}\right) \simeq \lim_{\omega\to 0}\left(\frac{\eta_0^2\omega^2}{G'}\right).$$

For a single Maxwell element, $E_0 = G_\infty$, but this is not true for a distribution of such elements.

Figure 2 shows a comparison of the shear stress in continuous shear (points surrounded by circles or squares) and the imaginary part G'' of the complex shear modulus, recalculated from Russell's compliance data in oscillatory shear. In figure 3 the same results are shown in the form of a comparison of the variation of the effective viscosity (P'_{12}/D or G''/ω) with shear rate or with angular frequency. On the basis of the considerations given above, the ratio η'/η_0 in continuous shear at a shear rate D should be the same as that in oscillatory shear at an angular frequency $\omega = KD$, where K is an unknown constant. It is seen from figure 3 that this is approximately correct over a narrow range of η'/η_0, between 1·0 and approximately 0·5, with K rather less than unity, approximately two-thirds. At higher frequencies and shear rates, however, the two sets of results diverge, and the ratio between them may amount to one or two orders of magnitude. It is seen from figures 1 and 2 that the two sets of results begin to diverge appreciably when the shear stress reaches a value of the order of E_0, i.e. of the order of a representative shear modulus. This is what we should expect from objection (iii) above (§3(c)), that the elasticity theory was valid only for small strains.

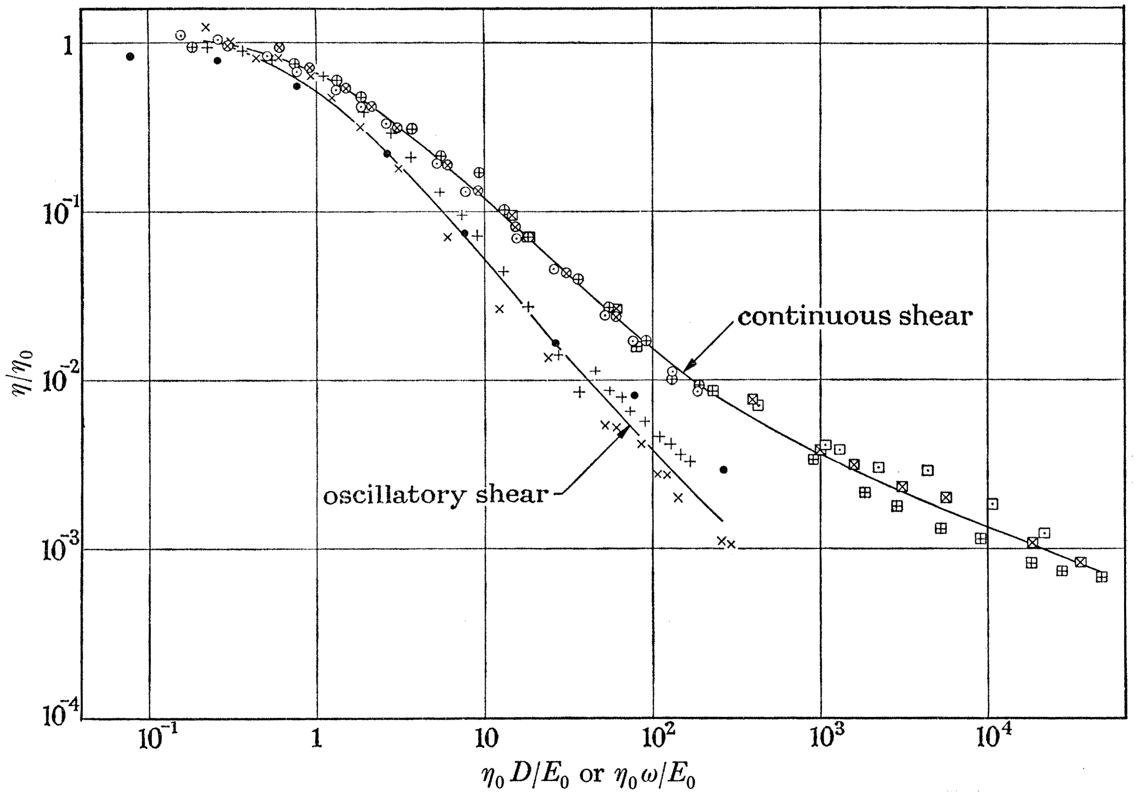

FIGURE 3. Comparison of variation of apparent viscosity in oscillatory and in continuous shear (results of Russell). (Results of figure 2 replotted.)

		AFGO/H	AFGO/L	SLW.10
oscillatory shear, η/η_0 against $\eta_0\omega/E_0$		×	●	+
continuous shear, η/η_0 against $\eta_0 D/E_0$	disks	⊗	⊙	⊕
	capillaries	⊠	⊡	⊞

It will also be seen from figure 2 that although the imaginary part G'' of the complex shear modulus passes through a maximum as $(\eta_0\omega)$ is increased, over the same region the shear stress maintains an approximately constant value and even continues to increase. The points surrounded by a square in figures 2 and 3 represent the results of capillary tube experiments, which Russell regards as less reliable than those of the disk experiments, represented by the circled points. Thus the sharp upturn of the shear stress curve in figure 2, and the change of slope in figure 3, should be accepted with some reserve.

In figure 4, the normal stress difference $(P_{11}-P_{22})$ in continuous shear is compared with the real part G' of the complex shear modulus in oscilltory shear. The comparison is seen to be approximately in accordance with the predictions of the theory over quite a wide range at low frequencies and shear rates, with K again approximately equal to unity. Again, the two sets of results diverge when the stresses approach the representative modulus E_0, and at high shear rates the normal stress differences may be up to an order of magnitude greater than those predicted from the oscillatory shear results.

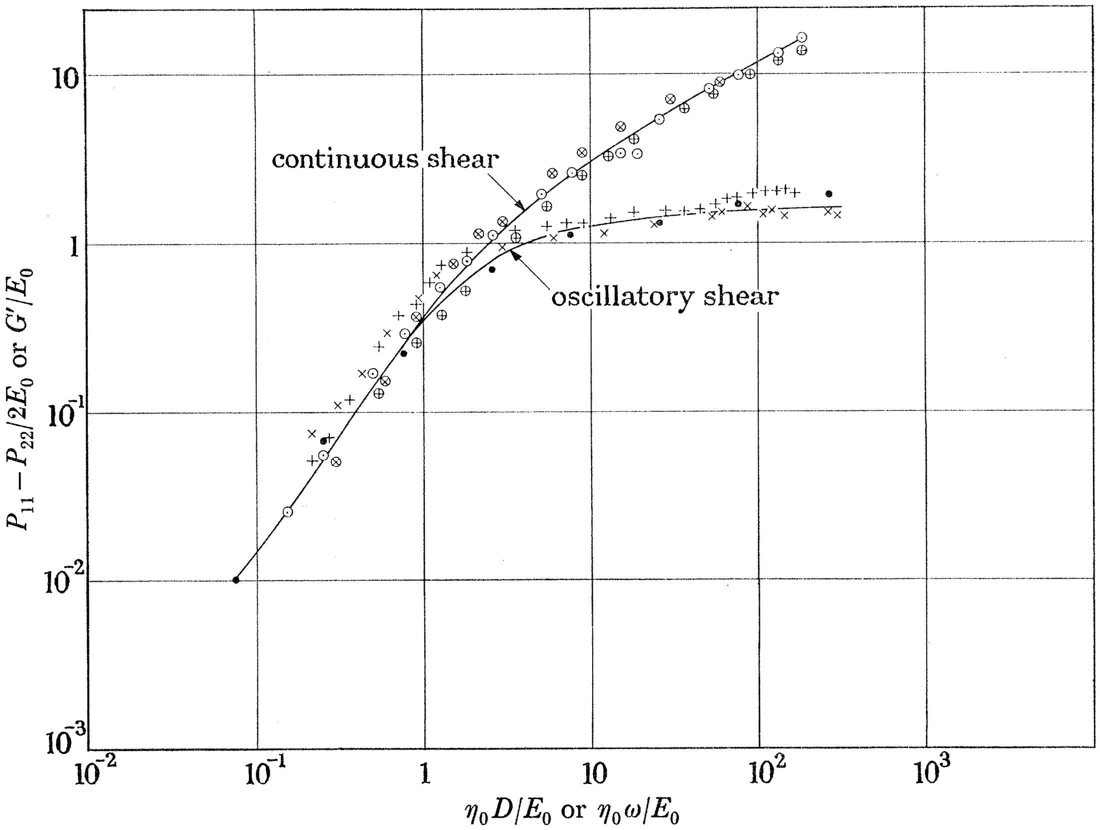

FIGURE 4. Comparison of normal stress difference in continuous shear with real modulus in oscillatory shear (results of Russell).

	AFGO/H	AFGO/L	SLW.10
oscillatory shear, G'/E_0 against $\eta_0\omega/E_0$	×	●	+
continuous shear, $P_{11}-P_{22}/2E_0$ against $\eta_0 D/E_0$	⊗	⊙	⊕

These conclusions from Russell's results are in agreement with those reported in the literature and mentioned above. Many authors have commented on the fact that the viscosities in continuous shear become higher than those in oscillatory shear at high frequencies and shear rates, even though agreement is found at moderate frequencies and shear rates. Cox & Merz (1958) proposed empirically that the effective viscosity in continuous shear should be compared not with the effective viscosity $\eta' = G''/\omega$ in oscillatory shear, but with the modulus of the complex shear viscosity:

$$|\eta^*| = \frac{|G^*|}{\omega} = \frac{(G''^2+G'^2)^{\frac{1}{2}}}{\omega} = \eta'\left[1+\left(\frac{G'}{G''}\right)^2\right]^{\frac{1}{2}}.$$

Using the results of de Witt *et al.* (1955), Cox & Merz showed that the value of $|\eta^*|$ in oscillatory shear agreed very well with the effective viscosity in continuous shear, with ω replaced directly by D, i.e. $K = 1$. This relation also seems to fit Russell's results very well indeed. Furthermore, the real component G' of the complex shear modulus, multiplied by the correction factor $(|\eta^*|/\eta')$, seems to agree with one-half of the normal stress difference over a much wider range than does the uncorrected component G' alone.

In the following section, we shall use the relation between the behaviour of viscoelastic substances in oscillatory and in continuous shear to correlate the various experimental measurements of the shear stress-shear rate relation for mineral oils under various conditions, and to relate them to their behaviour in oscillatory shear. The correction factor $(|\eta^*|/\eta')$ appears to be entirely empirical in character, and furthermore we shall be dealing with values of G'' on the rising branch of the curve, below the maximum. In this range the correction factor is approximately unity, to the degree of accuracy to which we shall be working. We shall therefore ignore the correction factor, and compare the shear stresses directly with values of G''.

In the comparisons between the behaviour of viscoelastic materials in oscillatory shear and that in continuous shear which various authors have made, attention is usually concentrated on the shear stress or viscosity variation, and the normal stress comparison seems to have been comparatively neglected. However, the normal stress comparison seems to be a more sensitive test of the viscoelastic theory than is the viscosity comparison, since the shear stress and G'' must coincide at low shear rates and low frequencies whatever the model used. On the other hand, there is no *a priori* reason to expect a relation between P'_{11} and G', other than the theory given above, and the range over which the relation may be tested with significant results is greater than in the case of the viscosity variations.

From the results of Barlow & Lamb (1959) we can calculate the value of E_0 for mineral oils as approximately 10^9 dyn/cm^2. In the elastohydrodynamic lubrication of sliding contacts we are interested in low shear rate viscosities of 10^3 to 10^6 P and shear rates of 10^6 to 10^8 s^{-1}. The dimensionless parameter $\eta_0 D/E_0$ therefore ranges from about unity to 10^5, i.e. approximately within the region covered by figures 2 and 3. We may therefore expect the theoretical relation between behaviour in continuous and in oscillatory shear, which we have shown is valid over a certain region for the materials studied by Russell, to be valid also in elastohydrodynamic lubrication, and we shall now enquire whether this expectation is correct or not.

4. COMPARISON OF THE BEHAVIOUR OF MINERAL OILS IN ELASTOHYDRODYNAMIC LUBRICATION WITH THEIR BEHAVIOUR IN OSCILLATORY SHEAR

(*a*) *General*

We have seen that there are grounds for supposing that the shear stress P_{12} in continuous shear at a shear rate D should be $1/K$ of the imaginary part G'' of the complex shear modulus in oscillatory shear, at an angular frequency $\omega = KD$, where K is an unknown constant. We also obtained a relation between the normal stress difference and the real part of the complex shear modulus, but normal stress differences under elastohydrodynamic conditions have not been measured, and we must work with the shear stresses.

The results in oscillatory shear are those of Barlow & Lamb (1959), who worked with three mineral oils of widely different viscosity index and constitution. The curves of G'' against $\eta_0\omega$ are very similar for these three oils, where η_0 is the viscosity at low frequency, and at the temperature and pressure considered.

The results in elastohydrodynamic lubrication were obtained on different oils, but because of the similarity of the results of Barlow & Lamb for the three different oils they used, we may hope that the comparison will not be vitiated for this reason. The limitations of the data derived from elastohydrodynamic lubrication have already been mentioned: the oil is subject to a pressure distribution which must be approximately Hertzian, but the detailed distribution is not known. However, the effective viscosity of the oil in an elastohydrodynamic contact is dominated by that at the maximum pressure (Crook 1963) and this cannot be very different from the Hertzian value.

To denote the fact that we are dealing with the mean resulting from a distribution of pressures, we shall use the symbol $\bar{\eta}'$ for the mean effective viscosity, i.e. that viscosity which would give the observed shear stress if it applied to all the lubricant in the Hertzian region. The symbol $\bar{\eta}_0$ will be used for the value of this viscosity at low rates of shear. This viscosity is defined by the slope of the initial linear portion of the curve of shear stress against shear rate.

It will be assumed that the frictional or shear forces were generated hydrodynamically, i.e. that boundary lubrication effects did not contribute significantly to the observed friction. This is probably true in Crook's work, where the lubricant films were at least 1 μm in thickness, while the centre-line-average surface roughness was of the order of 0·05 μm. The assumption may be less valid for Smith's experiments in which the lubricant films were sometimes much thinner. However, the analysis of Smith's results given below reveals the same general features as does that of Crook's work, and the neglect of boundary lubrication effects is therefore felt to be justified.

As the shear rate is increased, the shear stress in general reaches a maximum and then decreases. So far as is known no evidence of a decrease in shear stress with increasing shear rate has been shown under isothermal conditions, and it has been suggested that such a phenomenon would give an unstable flow pattern (Fromm 1948; Smith 1960). It will therefore be assumed that the decrease of shear stress with increasing shear rate is due to thermal effects and that there is a monotonically increasing functional relation between the shear rate and the shear stress at any given temperature. It is assumed that this is of the form

$$D = (\tau/\eta_0)f(\tau) \tag{7}$$

and the object is to recover $f(\tau)$, which is the reciprocal of the ratio η'/η_0 under isothermal conditions, from the experimental data. It will be assumed as an approximation that $f(\tau)$ is independent of temperature. We shall see later that this is not strictly true, but the variation of $f(\tau)$ with temperature is normally much less than the variation of η_0 with temperature, and the results should be approximately correct over small temperature ranges.

A theoretical analysis of the temperatures developed within an oil film with parallel bounding surfaces subject to shear is given in Appendix B. The general case is rather complex, but considerable simplification is possible if an exponential variation of viscosity with temperature may be assumed over a narrow temperature range;

$$\eta_0 = \alpha e^{-\beta\theta}, \tag{8}$$

where θ is the temperature.

Figure 5 shows an experimental curve of shear stress against shear rate, taken from Crook (1963), and the correction of the experimental results for thermal effects on the above basis. The corrections are explained in greater detail in appendix C, and in general the effect is to terminate the experimental curve at a point at or slightly below the shear stress maximum on the rising branch, the rest of the curve remaining substantially unaltered in most cases.

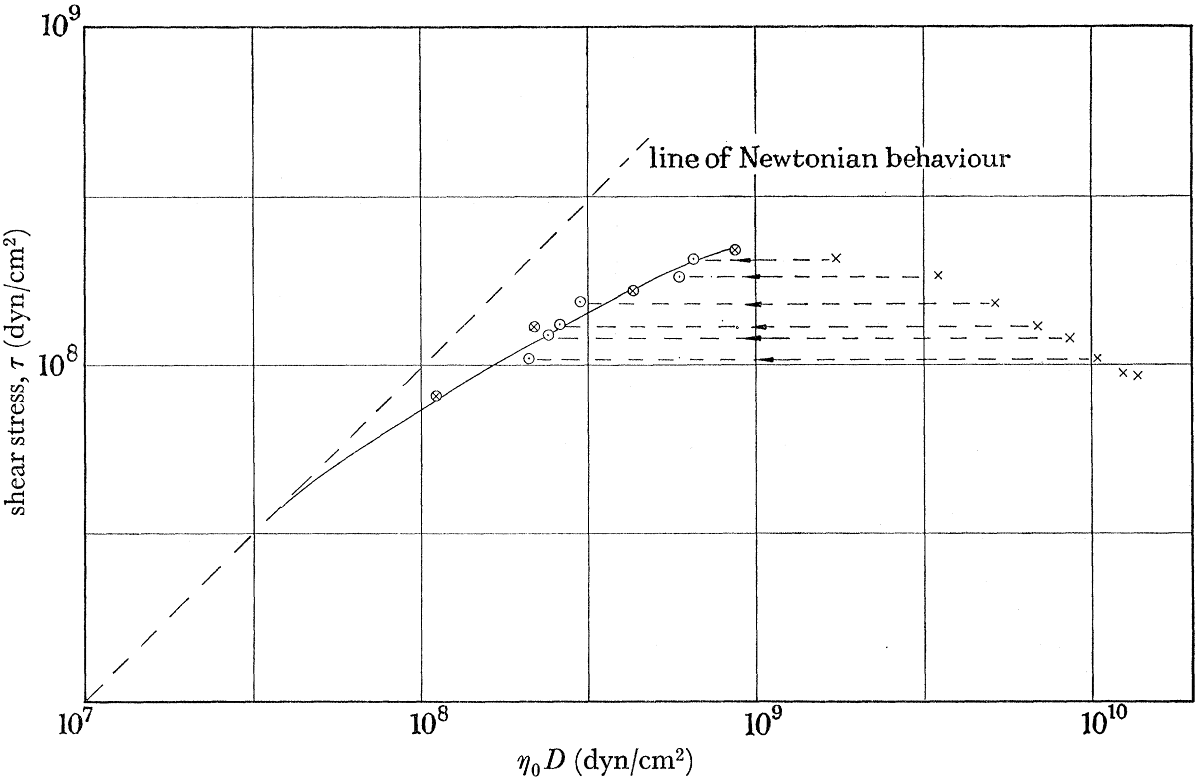

FIGURE 5. Example of temperature correction applied to an experimental curve of shear stress against shear rate. ×, Points read from experimental curve (Crook 1963); ⊙, corrected points; ⊗, points not needing correction.

(b) *Results of Crook* (1963)

In these experiments, pairs of steel disks were loaded together and the surface temperature was maintained constant. As the sum (U_1+U_2) of the velocities U_1 and U_2 of the surfaces of the disks, relative to the contact between them, was also constant, this means that the film thickness which was estimated from the electrical capacity between the disks, was also constant. The force of friction was measured as the sliding speed (U_2-U_1) was varied.

Figure 6 shows a comparison between the results of Crook in continuous shear and those of Barlow & Lamb in oscillatory shear. The curves of G'' against $\eta_0\omega$ are the results of Barlow & Lamb for the three mineral oils they used. The points represent $K\tau$ against $K\eta_0 D$, calculated from points read from the experimental curves of η' against U_2-U_1 given in figures 22 and 25 of Crook (1963) and corrected for thermal effects as explained above. Here τ is the shear stress, which has been calculated from the effective viscosity η' and the film thickness, D is the shear rate, calculated from the sliding velocity and the film thickness, and η_0 is the viscosity at low rates of shear. The constant K has been chosen for each set of points from Crook's results to bring them into approximate conformity with the curves of Barlow & Lamb. This may be done only by shifting the experimental curve of logarithm of shear stress against logarithm of $(\eta_0 D)$ in a direction bisecting the axes. By a suitable choice of the K's the results of Crook in continuous shear may be made to conform with those of Barlow & Lamb in oscillatory shear, as we have suggested above on theoretical grounds, and as we have seen is the case for other viscoelastic substances. However, the values of the constant K vary over the range from 4 to 12, in contrast to those of 0·4 to 1·0 which we encountered for the other viscoelastic substances. A table of values of K is given in appendix D. The shift constant K seems to decrease with increasing Hertz pressure, but there is insufficient evidence of the effect of temperature, which varied over a comparatively narrow range. A decrease in K means that deviations from Newtonian behaviour occur at higher shear stresses.

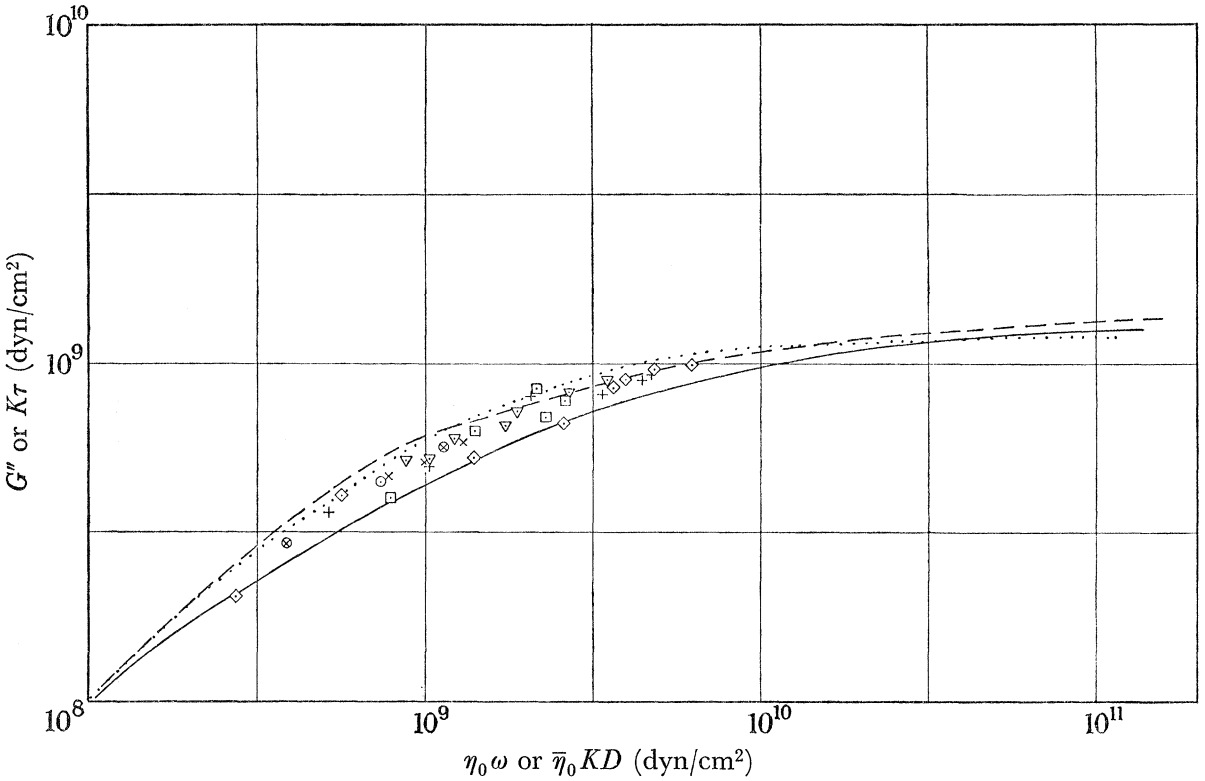

FIGURE 6. Comparison of results of Barlow & Lamb (1959) in oscillatory shear with those of Crook (1963) in continuous shear.

Oscillatory shear G'' against $\eta_0\omega$ results of Barlow & Lamb (1959): ······, LVI mineral oil; – – –, MVI mineral oil; ——, HVI mineral oil. Continuous shear $K\tau$ against $\eta_0 KD$ results of Crook (1963): +, fig. 22(*a*) in Crook (1963); × ⊙ ⊡, fig. 22(*b*) (symbols as in original); ▽, fig. 22(*c*); ◇, fig. 25.

(*c*) *Results of Smith* (1960)

Other results which we can use for our purpose are those of Smith (1960). These cover a much wider range of experimental conditions than do those of Crook, but there are a number of difficulties in their interpretation. These difficulties will now be discussed individually.

In the first place, the system is rather unusual and complex, in that a motor-driven cylindrical roller is in contact with a spherical roller which is free to rotate on a shaft, the axis of which may be placed at a skew angle relative to that of the other roller. In any one experimental run, the load and the peripheral speed U_1 of the motor-driven roller is maintained constant. The angle of skew, and therefore the relative sliding speed of the two surfaces, is varied, and the coefficient of friction estimated from the axial force on the spherical roller.

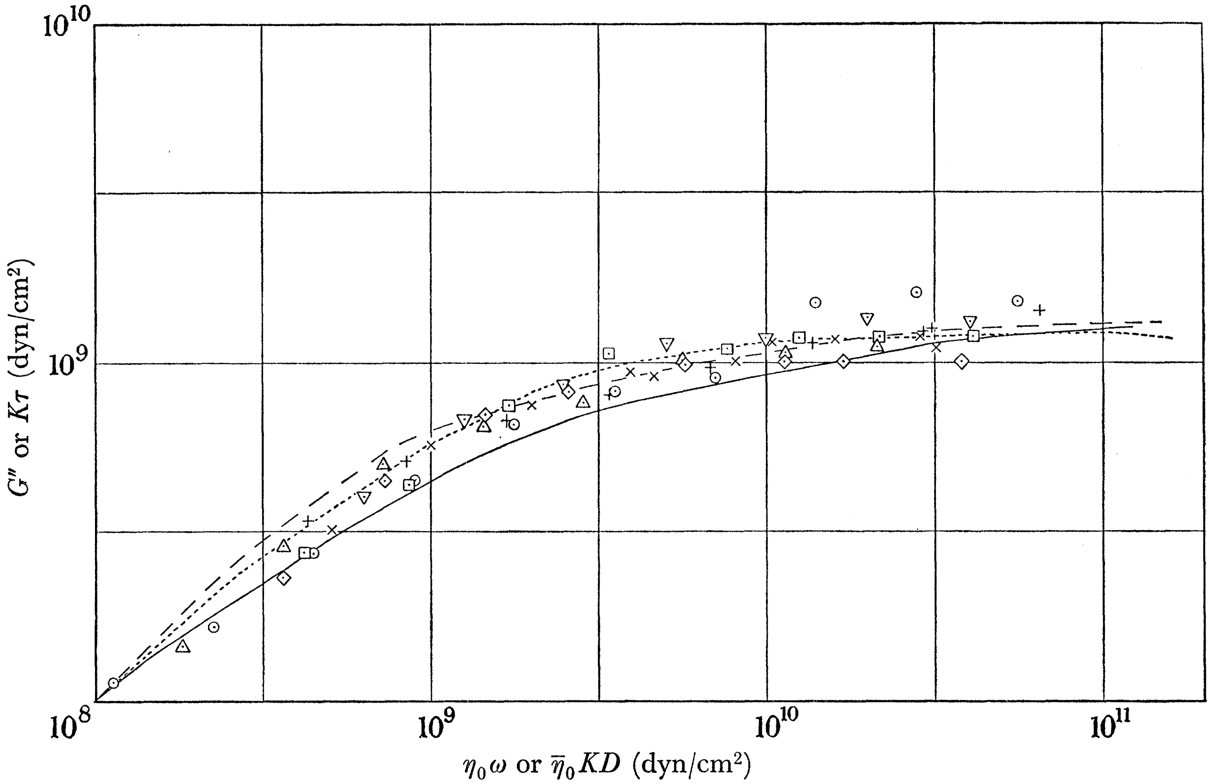

FIGURE 7. Comparison of results of Barlow & Lamb in oscillatory shear with those of Smith in continuous shear—mineral oils, steel surfaces.

Oscillatory shear G'' against $\eta_0\omega$ results of Barlow & Lamb (1959): ······, LVI mineral oil; – – –, MVI mineral oil; ——, HVI mineral oil.

Continuous shear $K\tau$ against $\eta_0 KD$ results of Smith (1960): (figure number in original reference):

	fig. 8		fig. 9		fig. 10
×	23 °C	▽	23 °C	⊡	25 °C
⊙	100 °C	+	100 °C	◇	100 °C
				△	190 °C

The conditions in the contact are thus rather complex. Smith assumes that the peripheral velocity of the spherical roller is equal to the component in that direction of the peripheral velocity of the cylindrical roller, but the results of Crook (1963) show that there must be a non-zero slip velocity. In the analysis of the results, we shall assume that this slip velocity is small compared with the rolling speed U_1.

Another difficulty is that the film thicknesses are not measured. We have therefore calculated them from the approximate theory of Grubin (1949), but for this purpose we need to know the temperatures of the roller surfaces. Smith measured his temperatures at a thermocouple placed in air near the point of contact, and we shall assume that the roller surface temperature was approximately equal to that registered on this thermocouple. The conditions in the contact differed from those assumed in the Grubin analysis, but the approximation should be adequate for small angles of skew. The analysis is in general restricted to small angles of skew by the need to consider thermal effects, both in the oil film and on the surfaces. The film thicknesses estimated from the theory of Grubin and used in the present analysis are given in appendix D.

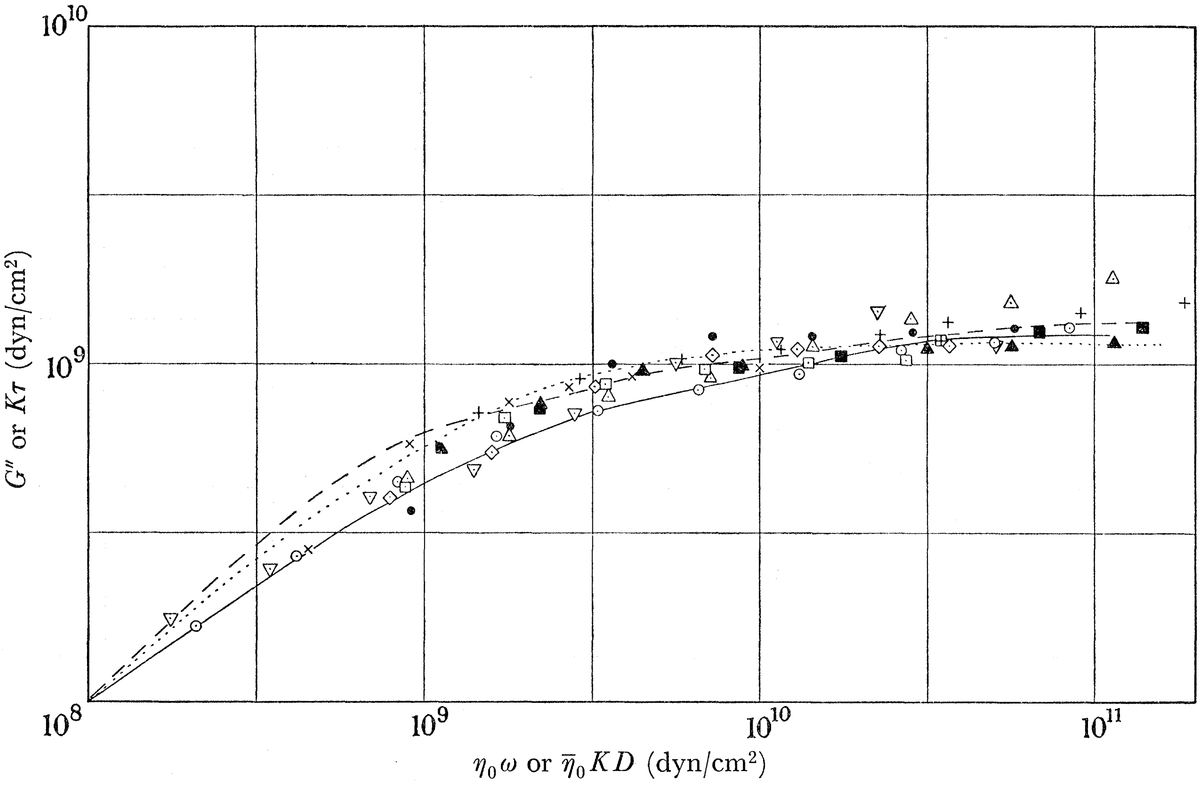

FIGURE 8. Comparison of results of Barlow & Lamb in oscillatory shear with those of Smith in continuous shear—mineral oils, tungsten carbide surfaces.

Oscillatory shear G'' against $\eta_0\omega$ results of Barlow & Lamb (1959): · · · ·, LVI mineral oil; – – –, MVI mineral oil; ——, HVI mineral oil.

Continuous shear $K\tau$ against $\eta_0 KD$ results of Smith (1960): (figure number in original reference):

fig. 12		fig. 13		fig. 14	
×	23 °C	⊡	23 °C	◇	23 °C $\bar{p} = 2\cdot80\times10^{10}$ dyn/cm^2
⊙	100 °C	+	100 °C	●	23 °C $\bar{p} = 3\cdot52\times10^{10}$ dyn/cm^2
▽	190 °C	△	190 °C	▲	100 °C $\bar{p} = 2\cdot80\times10^{10}$ dyn/cm^2
				■	190 °C $\bar{p} = 2\cdot80\times10^{10}$ dyn/cm^2

Another difficulty is that in most cases the curve of shear stress against shear rate does not cover a sufficiently wide range to include a linear portion at low shear rates. The effective viscosity $\bar{\eta}_0$ at low rates of shear must then be regarded as another disposable parameter, in addition to the factor K, available for shifting the experimental curve of shear stress against shear rate to conform with the results of Barlow & Lamb. The evidence of such agreement as is observed between the two sets of results is therefore less convincing than in the case of Crook's results. The values derived for $\overline{\eta}_0$ are included in the tables in appendix D.

Smith's oil was less viscous than Crook's, particularly at the two higher temperatures used, and his oil films were, in general, much thinner. The temperatures developed within the oil film are therefore less important compared with the flash temperature than is the case in Crook's work. Experimental results have been discarded if the flash temperature is excessive, and where necessary, corrections for thermal effects in the oil film have been made on the basis already examined. In many cases the complete experimental curve could be used without corrections.

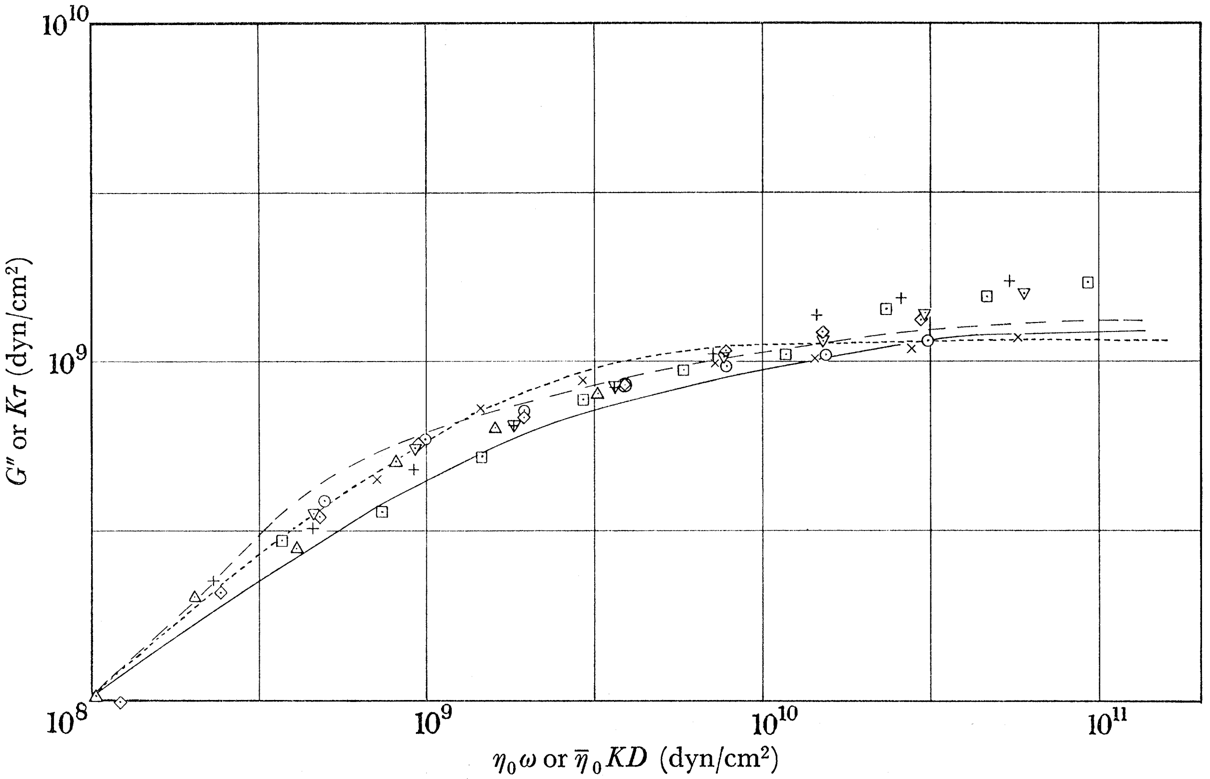

FIGURE 9. Comparison of results of Barlow & Lamb in oscillatory shear with those of Smith in continuous shear—mineral oils, steel against aluminium.

Oscillatory shear G'' against $\eta_0 D$ results of Barlow & Lamb (1959): - - - -, LVI mineral oil; – – –, MVI mineral oil; ——, HVI mineral oil. Continuous shear $K\tau$ against $\eta_0 KD$ results of Smith (1960): (figs. 5 and 6 in original reference). Rolling speeds (cm/s): ×, 8; ⊙, 16; ▽, 32; ⊡, 64; +, 128; ◇, 256; △, 512.

The results of Smith in continuous shear are compared with those of Barlow & Lamb in figures 7, 8 and 9. These show rather more scatter than that in figure 6, but this was only to be expected in view of the difficulties mentioned above. The shift factors K ranged from 0·4 to 8, and a table of these values is given in appendix D.

(d) *Variation of shift factor K with conditions of experiment*

The variation of the shift factor K with temperature and pressure is shown in figure 10. The line marked 'hypothetical asymptote at high pressures' will be explained in the Discussion below. The results both of Crook and of Smith indicate that K decreases as the

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Hertzian pressure increases, i.e. significant departure from Newtonian behaviour is postponed to higher shear stresses but occurs at lower shear rates. Furthermore, the results of Smith show that K increases as the temperature increases, i.e. significant departure from Newtonian behaviour occurs at lower shear stresses, but at higher shear rates. Hence the function (τ) used in the correction for thermal effects in the oil film (§4 (*a*) above) varies with temperature, but the variation is small compared with that of η_0.

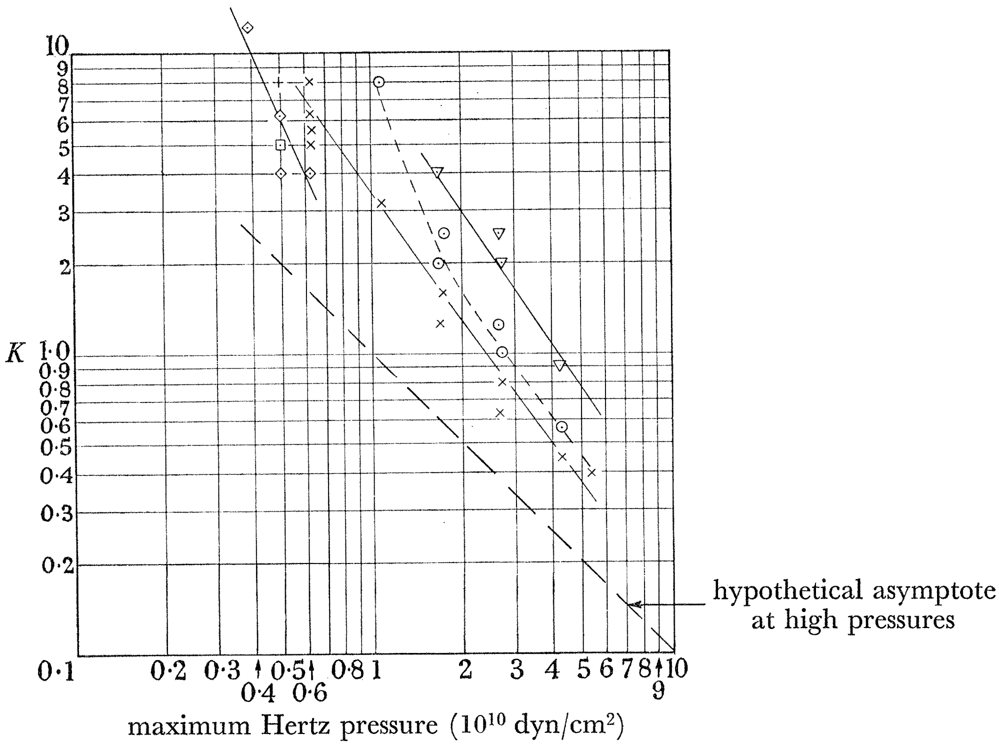

FIGURE 10. Variation of shift factor K with pressure and temperature. Crook (1963): ×, 25 °C; □, 30 °C; ◇, 50 °C. Smith (1960): ×, 23 °C; ⊙, 100 °C; ▽, 190 °C.

In its dependence on temperature and pressure, the behaviour of the shift factor K is qualitatively similar to that of the fluidity (reciprocal of viscosity). However, K seems to be approximately independent of rolling speed, which has an unexpectedly large influence on the effective viscosity $\bar{\eta}_0$. It also seems to be independent of the material of the rollers.

The variation of effective viscosity with rolling speed raises an interesting point. Crook (1963) was the first to point out that the effective viscosity $\bar{\eta}_0$ of a mineral oil, under elastohydrodynamic conditions and at very low sliding speeds, decreased rapidly as the rolling speed increased. The same conclusion may be derived from Smith's results, particularly those of figures 5 and 6 of Smith (1960). Here the effective viscosity $\bar{\eta}_0$ decreased by a factor of approximately 20, from $1{\cdot}3 \times 10^5$ P at a rolling speed of 8 cm/s to $5{\cdot}4 \times 10^3$ P at 512 cm/s. Over this range the value of K did not change significantly (appendix D, table 5). These estimates are subject to the uncertainty of the film thickness calculations, but it seems unlikely that the relative error involved could exceed a factor of, say, 3.

Crook showed that this decrease in viscosity with increasing rolling speed could not be explained by thermal effects arising from compression of the oil or from shear in the inlet region. He explained his observations on the basis of a Maxwell model, but he found that he had to assume values of G_∞ as low as $1{\cdot}4\times10^6$ dyn/cm², which is much lower than the value of G'_∞ of approximately 10^{10} dyn/cm² reported by Barlow & Lamb (1959).

It therefore appears unlikely that this is the explanation of the effect observed by Crook. This is confirmed by an application of the principles discussed in § 3(*c*) above, where it was seen that non-Newtonian behaviour of a Maxwell substance became important at much lower shear rates than would be expected from a straightforward application of the Maxwell equation.

From equations (2) of § 3 (*c*) it is seen that the variation of effective viscosity with shear rate is such that the effective viscosity η' falls to one half of its initial value η_0 at a shear rate of the order of the reciprocal of the relaxation time, i.e. approximately $10^4\,\mathrm{s}^{-1}$ in the cases reported by Crook. The thickness of the lubricant film is approximately 10^{-4} cm, and this shear rate therefore corresponds to a sliding velocity of approximately 1 cm/s. It is apparent from the results given by Crook that the variation of effective viscosity with sliding speed is very small up to sliding speeds of well beyond 1 cm/s. Thus the explanation in terms of a Maxwell model of the variation of the effective viscosity with rolling speed at low sliding speeds is inconsistent with the observed variation of effective viscosity with sliding speed at constant rolling speed. This effect remains one of the major unsolved mysteries of elastohydrodynamic lubrication, and until we have explained it we cannot claim to understand the physics of the subject.

5. Discussion

(*a*) *General*

We have seen that, on certain assumptions, there are theoretical grounds for supposing that the variation of the effective viscosity η' of a viscoelastic substance with shear rate D in continuous shear will be the same as that with angular frequency ω in oscillatory shear, with $\omega = KD$ and K is an unknown constant. In other words, the shear stress τ in continuous shear is $1/K$ of the imaginary component G'' of the complex shear modulus in oscillatory shear at a frequency $\omega = KD$. Furthermore, the normal shear stress difference $(P_{11}-P_{22})$ in continuous shear should be $2/K^2$ of the real part G' of the complex shear modulus, again at an angular frequency $\omega = KD$.

We have seen that both these predictions are confirmed for several solutions which exhibit viscoelastic behaviour at atmospheric pressure and at low rates of shear. For mineral oils under elastohydrodynamic conditions we have confirmed the shear stress behaviour, but we have no means of investigating the normal stress differences.

(*b*) *Normal stress differences*

From the values of K derived from the shear stresses, we may estimate the order of magnitude of the normal stress differences, on the basis of viscoelastic theory and of the results of Barlow & Lamb in oscillatory shear. We find normal stress differences of up to approximately 10^{10} dyn/cm² (150 000 Lb./in.²) exerted as a tension along the stream lines. This would be a favourable effect, i.e. it would increase the film thickness under given conditions, since the tension would increase in the inlet region as the oil approached the Hertzian zone. Thus the pressure gradient opposing the flow of oil would be less, the flow rate would be greater, therefore the film thickness at the position of zero velocity gradient would be greater. However, it is very difficult to give a quantitative estimate of the effect. The hydrodynamic theory of viscoelastic materials is very complicated, particularly if they are regarded as compressible. In elastohydrodynamic theory, we must take into account the variation of viscosity with pressure, which is closely related to the variation of volume with pressure. If there are normal stress differences, the volume is not simply related to the stress pattern, and the viscosity would be expected to behave in an equally complicated manner. We therefore ignore the normal stress differences and concentrate our attention on the shear stresses.

(*c*) *Shear stresses*

By the procedure outlined above, we may reduce an apparently arbitrary collection of curves of shear stress against shear rate, obtained under different experimental conditions, to one curve, which may be obtained from the behaviour in oscillatory shear, together with a shift factor K for each experimental condition. We now confront the problem of reducing the variation of K with experimental conditions to a rational basis. Unfortunately our data depend on the analysis of the results of Smith, and the difficulties in the interpretation of these for our purposes have already been explained. We now attempt an interpretation of these results on the basis of the properties of the Oldroyd model, but this must remain speculative until further experimental evidence can be obtained.

We imagine the oil as being composed of a number of Oldroyd elements, simplified as discussed above, each characterized by a relaxation time λ_1 and another parameter μ_0, also with the dimensions of time. In oscillatory shear, only the parameter λ_1 enters, and Barlow & Lamb (1959) show that when conditions of temperature and pressure are changed, the value of λ_1 for each element changes in the same ratio, that of the viscosity η_0 at low frequency under the appropriate conditions of temperature and pressure, i.e.

$$(\lambda_1)_2/(\lambda_1)_1 = (\eta_0)_2/(\eta_0)_1. \tag{9}$$

In continuous shear we must consider also the effect of the parameter μ_0. The agreement which we have noted above between the shapes of the curves of G' against $\eta_0\omega$ in oscillatory shear and of τ against $\eta_0 D$ in continuous shear suggests that the ratio $K^2 = \mu_0/\lambda_1$ is the same for each element. This ratio changes with pressure and temperature, but we may conclude that the value of μ_0 for any one particular element will change in the ratio

$$(\lambda_1 K^2)_2/(\lambda_1 K^2)_1 = (\eta_0 K^2)_2/(\eta_0 K^2)_1$$

from equation (9). In these equations, the subscripts 1 and 2 outside the brackets refer to the conditions of temperature and pressure.

A representative value of the parameter μ_0 is therefore given by the expression $(\bar{\eta}_0 K^2)$. It is shown in appendix E, from an analysis of some of the results of Smith, that this expression is approximately independent of pressure, but that it varies with temperature in a manner such that it is approximately proportional to the viscosity $(\eta_0)_0$ at low rates of shear at atmospheric pressure.

Thus $$\frac{\bar{\eta}_0 K^2}{(\eta_0)_0} = \text{constant}, \simeq 10^6. \tag{10}$$

The value of this constant may be expected to vary with factors such as the rolling speed, the nature of the oil, etc.

This relation would imply $K \simeq 10^3$ at atmospheric pressure, whereas the materials used by other workers give $K \simeq 1$. This may reflect the different mechanisms involved in the viscoelastic effects. For these other substances, $G'_\infty \simeq 10^3\,\text{dyn/cm}^2$, and the viscous and elastic effects may plausibly be attributed to the formation on a three-dimensional network of long chains, with junctions between the molecular chains of varying degrees of permanence (Lodge 1956). This mechanism is untenable for mineral oils, with $G'_\infty \simeq 10^{10}\,\text{dyn/cm}^2$, as the concentration of chain segments required would exceed one per carbon atom.

The suggestion that $K \simeq 10^3$ for mineral oils at atmospheric pressure receives some support from the results of Porter & Johnson (1959), who used a rotational viscometer. From their results it is possible to select five mineral oils whose effective viscosities η' have been measured down to $\eta'/\eta_0 \simeq 0{\cdot}8$ at 100 °F. Unfortunately the results for the three oils used by Barlow & Lamb (1959) show rather more scatter at these higher values of η'/η_0, and this viscosity ratio is observed in oscillatory shear at values of $(\eta_0\omega)$ between 2×10^8 and $6\times10^8\,\text{dyn/cm}^2$, with a geometric mean of $4\times10^8\,\text{dyn cm}^{-2}$. The five oils used by Porter & Johnson gave a ratio of η'/η_0 of $0{\cdot}8$ at values of $(\eta_0 D)$ ranging from $1{\cdot}6\times10^5$ to $5{\cdot}4\times10^5\,\text{dyn/cm}^2$, with a geometric mean of $3\times10^5\,\text{dyn/cm}^2$. The mean value of K was therefore approximately $1{\cdot}3\times10^3$. Mineral oils have been reported to show non-Newtonian behaviour in the capillary jet viscometer at considerably lower shear rates (Morris & Schnurmann 1946, 1951; Schnurmann 1962). However, the jet viscometer data have recently been reviewed by Porter & Johnson (1962) who concluded that the jet viscometer results may be in error, possibly owing to capillary end effects and kinetic energy corrections. It seems, therefore, that equation (10) is supported to some extent by measurements at atmospheric pressure, which agree approximately with the conclusion obtained by extrapolation of the results we have obtained from the field of elastohydrodynamic lubrication.

(*d*) *Extrapolation of shear stresses to higher shear rates*

The oscillatory shear results given by Barlow & Lamb show a maximum in G'' as $(\eta_0\omega)$ increases, and G'' tends to zero as $(\eta_0\omega)$ tends to infinity. By analogy with Russell's results we should not expect this maximum to be reproduced in the shear stress curves under isothermal conditions. Instead, we should expect the shear stress to remain approximately constant at values of $\eta_0 KD$ higher than those shown in figures 6 to 9, and a suitable equation representing such behaviour would be

$$\tau = \eta_0 D\left(1+\frac{\eta_0 K|D|}{G''_{\text{max.}}}\right)^{-1}, \tag{11}$$

where $G''_{\text{max.}} \simeq 1{\cdot}3\times10^9\,\text{dyn/cm}^2$ and K may be obtained approximately from equation (10). Equation (11) satisfies the condition that $(K\tau)$ should be a function of $(\eta_0 KD)$, this function being independent of pressure, temperature, etc., and representing the variation of G'' with $(\eta_0\omega)$ in the ascending part of the curve. The accuracy of this representation is shown in figure 11.

In practice, thermal effects would cause a decrease in shear stress with increasing shear rate. Thus the curves both of G'' against frequency and of shear stress against shear rate would show a maximum, but for different reasons.

The effect of temperature variations within the oil film may be investigated by the method given in appendix B, with

$$f(\tau) = \left(1 - \frac{K|\tau|}{G''_{\text{max.}}}\right)^{-1}.$$

Here we have to assume that K is approximately independent of temperature. This approximation is adequate only over a narrow temperature range.

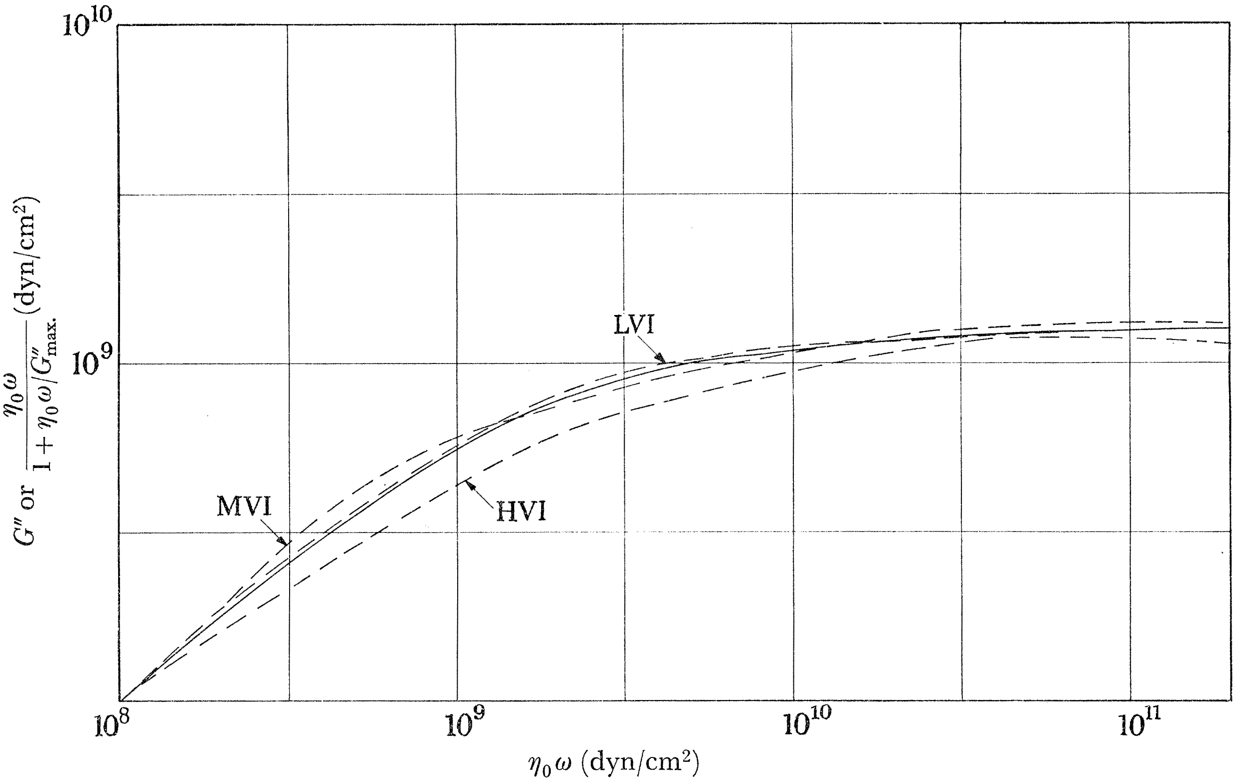

FIGURE 11. Comparison of experimental results in oscillatory shear with an empirical formula. — —, Experimental results G'' (Barlow & Lamb 1959); ———, $\frac{\eta_0\omega}{1+\eta_0\omega/G''_{\text{max,}}}$.

(e) *Extrapolation to boundary lubrication conditions*

We have suggested equation (10) as an empirical description of the results of our analysis of the non-Newtonian behaviour of Smith's oil under the range of conditions of his experiment. We have tentatively extrapolated this equation to atmospheric pressure, but we would not expect to be able to extrapolate it to very high pressures, for then η_0 tends to infinity, K would tend to zero, and the maximum shear stress would tend to infinity.

From equation (11), the maximum shear stress is

$$\tau_{\text{max.}} = G''_{\text{max.}}/K. \tag{12}$$

For very high pressures and very thin films, we should expect the result characteristic of boundary lubrication,

$$\tau = f_b p, \tag{13}$$

where f_b is the coefficient of boundary friction. Now consider a very thin oil film between opposing asperities of two surfaces. The pressure will be of the order of the hardness of the softer surface, and the viscosity at low rates of shear will be very high. The oil will be retained between the surfaces because of this high viscosity. If an increasing shear stress is applied to the surfaces, they will eventually be set in relative motion. Since the velocity is very low, the temperature of the lubricant within the film will not exceed that of the surfaces by any significant amount. Therefore the only mechanism available to limit the shear stress is the viscoelastic property of the lubricant, and we may equate τ in equation (13) with $\tau_{\text{max.}}$ in equation (12). This gives

$$K = \frac{G''_{\text{max.}}}{f_b p} \simeq \frac{10^{10}}{p}, \tag{14}$$

where p is in dyn/cm^2 and we have taken $G''_{\text{max.}} = 1{\cdot}3 \times 10^9$ dyn/cm^2, $f_b = 0{\cdot}13$.

The line labelled 'hypothetical asymptote at high pressures' in figure 10 has been drawn on the basis of equation (14), and should be independent of temperature. The experimental points seem to be consistent with the position of this asymptote, but further information is obviously desirable.

6. Conclusion

Departures from non-Newtonian behaviour exhibited by mineral oils in elastohydrodynamic lubrication are due to viscoelasticity. The model of a mineral oil which has been found necessary consists of a series of Oldroyd elements, each characterized by a relaxation time λ_1 and an additional parameter μ_0.

When the temperature and pressure of the environment change, e.g. from condition 1 to condition 2, each of the values of λ_1 associated with the individual elements changes in a ratio $(\lambda_1)_2/(\lambda_1)_1$ equal to the ratio $(\eta_0)_2/(\eta_0)_1$ of the viscosities of the real liquid at low rates of shear or low frequency, under the conditions considered. The values of μ_0 characterising each element change by a different constant factor which seems to be approximately independent of pressure. Its variation with temperature in the one case investigated is approximately the same as that of the viscosity $(\eta_0)_0$ at low rates of shear and at atmospheric pressure.

The oils used in the continuous shear work were different from those used in the oscillatory shear experiments; there is an obvious need for experiments of both types to be conducted with the same fluids. It would materially assist in the interpretation of the results if the experiments in continuous shear could be undertaken under simple known and defined conditions, e.g. in a rotational viscometer rather than in an elastohydrodynamic contact. This would greatly assist an attempt to further our knowledge of the physics of the elastohydrodynamic lubrication process.

Thanks are due to Professor J. Lamb for drawing the attention of the author to Dr Russell's thesis, and to Dr R. J. Russell and to the University of London for permission to quote some of the experimental results contained in that thesis. The author also wishes to thank many colleagues, particularly Dr H. Naylor, for helpful discussions.

APPENDIX A. APPLICABILITY OF THE MAXWELL EQUATION

We consider under what conditions we may apply equation (1) of the text to a fluid subject to simple shear, at a shear rate D. If the material is isotropic and incompressible, equation (1) must be considered as a special case of a general tensor equation, obtained by replacing D by the strain rate tensor, and τ by the stress tensor P'_{ik}. Now the tensors on the right-hand side of equation (1) of the text are symmetrical, that is, they have the property

$$P'_{ik} = P'_{ki}.$$

But in simple shear, the strain rate tensor is

$$\begin{pmatrix} 0 & \dfrac{\partial u_1}{\partial x_2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where u_1 is the velocity of a particle of the fluid in the '1' direction. This tensor is not symmetrical, and equation (1) cannot be satisfied as it stands. The situation may be rectified if the system of axes is rotated about the '3' direction. If the angular velocity of this rotation is Ω, the strain rate tensor in the new system becomes

$$\begin{pmatrix} 0 & \dfrac{\partial u_1}{\partial x_2}-\Omega & 0 \\ \Omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This is symmetrical if $\Omega = \frac{1}{2}(\partial u_1/\partial x_2) = \frac{1}{2}D$, where D is the shear rate. Before we can apply equation (1) of the text we must therefore use a system of axes which rotate relative to the planes of shear.

APPENDIX B. SHEAR STRESS AND TEMPERATURE RISE IN AN ELASTOHYDRODYNAMIC FILM WITH A NON-NEWTONIAN LUBRICANT

In figure 12 the origin is taken in the centre of the oil film and axes Ox, Oy respectively parallel and perpendicular to the direction of the film. The two parallel bounding surfaces of the film have velocities U_1 and U_2 in the x direction. We are concerned only with conditions in the essentially parallel portion of the film, since the pressure, viscosities and shear stresses outside this region are small compared with those inside the region. Within this region, the pressure gradient $\mathrm{d}p/\mathrm{d}x$ will be considered as zero.

We suppose that the relation between shear stress τ and shear rate D is of the form

$$D = \frac{\mathrm{d}u}{\mathrm{d}y} = \frac{\tau}{\eta_0} f(\tau), \tag{B1}$$

where η_0 is the viscosity at low shear rates, and $f(\tau)$ is a function which is independent of temperature.

We further suppose that the variation of η_0 with temperature θ is of the form

$$\eta_0 = \alpha \mathrm{e}^{-\beta\theta}. \tag{B2}$$

We assume that the temperatures θ_w of the two plane boundaries of the film are equal. Then, by symmetry, the temperatures will be equal at all points equidistant from the central plane of the film, but on opposite sides. The viscosities η_0, and therefore the velocity gradients D, will therefore be equal in magnitude, but the velocity gradients will be opposite in sign. The velocity u_0 of the oil in the central plane of the film will therefore be

$$u_0 = \tfrac{1}{2}(U_1+U_2). \tag{B 3}$$

The treatment may be generalized to include the case where the temperatures of the walls are different.

Consider a plane laminar element of fluid between the limits y and $(y+\delta y)$. Since $\mathrm{d}p/\mathrm{d}x = 0$, the shear stress τ is independent of y. The condition for thermal equilibrium of the element is

$$\tau\frac{\mathrm{d}u}{\mathrm{d}y}+k\frac{\mathrm{d}^2\theta}{\mathrm{d}y^2} = 0, \tag{B 4}$$

where u is the velocity of the fluid in the x direction and k is the thermal conductivity of the oil. It is assumed that all the heat is conducted to the walls which are isothermal, (Crook 1963), and that steady-state temperatures exist in the oil film (Archard, 1958/9).

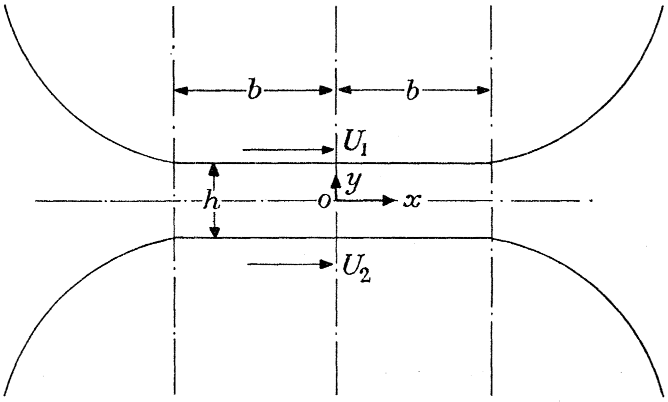

FIGURE 12. Model of elastohydrodynamic contact.

Equation (B 4) may be integrated to give

$$\tau u+k\,\mathrm{d}\theta/\mathrm{d}y = \text{constant} = \tfrac{1}{2}\tau(U_1+U_2)$$

from equation (B 3) since $\mathrm{d}\theta/\mathrm{d}y = 0$ in the central plane $y = 0$. At the walls,

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}y}\right)_w = \pm\frac{\tau}{k}\,\frac{U_2-U_1}{2}. \tag{B 5}$$

Equations (B 1) and (B 4) give

$$k\frac{\mathrm{d}^2\theta}{\mathrm{d}y^2}+\frac{\tau^2}{\eta_0}f(\tau) = 0.$$

This may be integrated after multiplication by $2\,\mathrm{d}\theta/\mathrm{d}y$ to give

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}y}\right)^2+\frac{2\tau^2}{k}f(\tau)\int_{\theta_0}^{\theta}\frac{\mathrm{d}\theta}{\eta_0} = 0, \tag{B 6}$$

where θ_0 is the maximum temperature in the oil film at $y = 0$. Application of equations (B 5) and (B 6) to conditions at the wall, gives

$$\int_{\theta_w}^{\theta_0}\frac{\mathrm{d}\theta}{\eta_0} = \frac{(U_2-U_1)^2}{8k}\,\frac{1}{f(\tau)}. \tag{B 7}$$

Equation (B 6) may be rewritten in the form

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = \frac{1}{\tau}\sqrt{\left(\frac{k}{2f(\tau)}\right)}\bigg/\sqrt{\int_{\theta}^{\theta_0}\frac{\mathrm{d}\theta}{\eta_0}}.$$

This equation may be integrated by the use of equation (B 2) to give

$$y = \frac{1}{\tau}\sqrt{\left(\frac{k\alpha\beta\,\mathrm{e}^{-\beta\theta_0}}{2f(\tau)}\right)}\int_{\theta_0}^{\theta}\frac{\mathrm{d}\theta}{\sqrt{(1-\mathrm{e}^{-\beta(\theta_0-\theta)})}}.$$

The integral may be evaluated by the substitution

$$u = \mathrm{e}^{\frac{1}{2}\beta(\theta_0-\theta)}$$

and application to conditions at the walls gives

$$\pm\frac{h}{2} = \frac{1}{\tau}\sqrt{\left(\frac{2k\alpha\,\mathrm{e}^{-\beta\theta_0}}{f(\tau)}\right)}\ln\{u_w+\sqrt{(u_w^2-1)}\}, \tag{B 8}$$

where

$$u_w = \mathrm{e}^{\frac{1}{2}\beta(\theta_0-\theta_w)} \tag{B 9}$$

and h is the oil film thickness.

Equations (B 7) and (B 2) give

$$\frac{(U_2-U_1)^2}{8k}\frac{1}{f(\tau)} = \frac{\mathrm{e}^{\beta\theta_0}-\mathrm{e}^{\beta\theta_w}}{\alpha\beta} = \frac{u_w^2-1}{\beta(\eta_0)_w}. \tag{B 10}$$

Also in equation (B 8)

$$\alpha\,\mathrm{e}^{-\beta\theta_0} = (\eta_0)_w/u_w^2, \tag{B 11}$$

where

$$(\eta_0)_w = \alpha\,\mathrm{e}^{-\beta\theta_w} \tag{B 12}$$

is the viscosity at low shear rates at the temperature θ_w of the walls.

Equations (B 8), (B 10) and (B 11) give

$$\frac{\tau f(\tau)}{(\eta_0)_w(U_2-U_1)/h} = \frac{\ln\{\epsilon+\sqrt{(1+\epsilon^2)}\}}{\epsilon\sqrt{(1+\epsilon^2)}}, \tag{B 13}$$

where

$$\epsilon^2 = \frac{(\eta_0)_w\beta(U_2-U_1)^2}{8k}\frac{1}{f(\tau)}. \tag{B 14}$$

From equations (B 13) and (B 14), τ may be derived if the other variables are known. From equations (B 10) and (B 14), $u_w^2 = 1+\epsilon^2$ and the maximum temperature rise $(\theta_0-\theta_w)$ may then be calculated from equation (B 9).

In the construction of figure 13, the abscissa is taken to be

$$X = (\eta_0)_w(U_2-U_1)^2\beta/8k$$

and the ordinate to be

$$Y = \eta'/(\eta_0)_w,$$

where η' is an effective viscosity defined by

$$\eta'(U_2-U_1)/h = \tau.$$

The parameter in figure 13 is

$$Z = 1/f(\tau).$$

Equations (B13) and (B14) then give

$$\frac{Y}{Z} = \frac{\ln\{\epsilon + \sqrt{(1+\epsilon^2)}\}}{1+\epsilon^2}, \tag{B15}$$

with

$$\epsilon^2 = XZ. \tag{B16}$$

For any pair of values of ϵ and Z, X is obtained from equation (B16) and Y from equation (B15). If a number of values of ϵ are taken and Z is maintained constant, a relation between

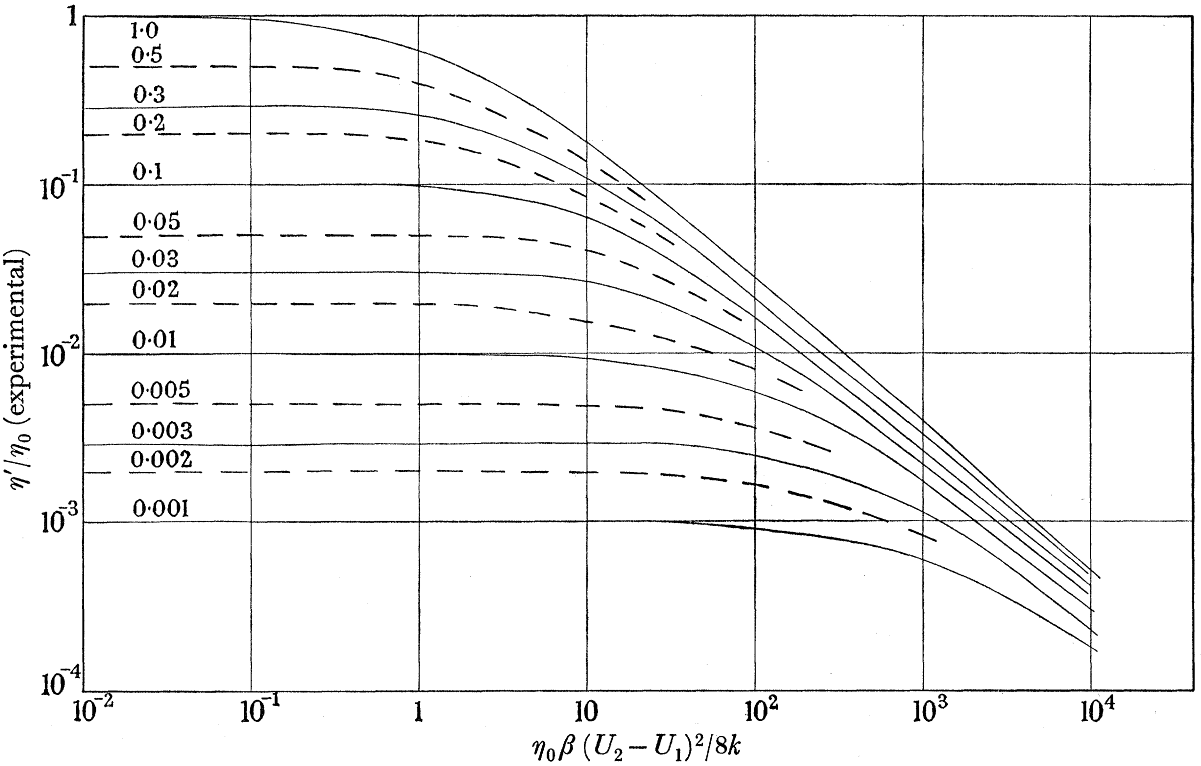

FIGURE 13. Temperature corrections for shear of non-Newtonian lubricant. The numbers against the curves refer to the quantity η'/η_0 (isothermal) or $1/f(\tau)$.

X and Y at constant Z is obtained. Such relations are shown in figure 13. The ordinate Y corresponds to the experimental ratio $\bar{\eta}'/\bar{\eta}_0$ and the parameter Z gives the ratio $\bar{\eta}/\bar{\eta}_0$ which would have been observed at the experimental value of the shear stress if isothermal conditions had prevailed.

APPENDIX C. CORRECTION OF EXPERIMENTAL SHEAR STRESS-SHEAR RATE CURVES FOR THERMAL EFFECTS

In figure 5, the points marked with a cross have been read off the experimental curve of effective viscosity against sliding speed, figure 22 (*c*) of Crook (1963). (They are not Crook's experimental points.) It is assumed that points on the right-hand side of the shear stress maximum would give the same value of $f(\tau)$ as a coresponding point at the same shear stress on the ascending part of the curve provided that the thermal effects at this latter point are negligible. Then the experimental value of $\overline{\eta}'/\overline{\eta}_0$ is entered on the ordinate of figure 13, and the parameter of the curves is set equal to the supposed value of $f(\tau)$ taken from the point on the rising part of figure 5. The abscissa of the point thus defined is

$$\overline{\eta}_0\beta(U_2-U_1)^2/8k$$

and from this a value of β/k may be calculated. This is done for all the points on the descending part of figure 5, starting with the point corresponding to the highest shear rate. For example, the following successive values of $10^6\beta/k$ ($\text{dyn}^{-1}\,\text{s}$) were obtained from the points shown in figure 5, starting with that with the highest shear rate

$$8{\cdot}1, \quad 8{\cdot}5, \quad 7{\cdot}7, \quad 8{\cdot}1.$$

The value of β/k was therefore taken to be $8{\cdot}1\times10^{-6}\,\text{dyn}^{-1}\,\text{s}$. This compares with the value of $5{\cdot}6\times10^{-6}\,\text{dyn}^{-1}\,\text{s}$ given in table 3 of Crook (1963). The experimental points were then corrected to isothermal conditions, with our values of β/k. Sometimes the calculated values of β/k increase at first as points with decreasing shear rate are selected. This reflects the fact that the exponential viscosity-temperature relation is valid only over a comparatively narrow range, and that the effective value of β decreases as the temperature range increases. However, the values normally tend to stabilize as the shear stress maximum is approached, and it is necessary to discard those points with values of β/k significantly lower than the stabilized value.

When the value of β/k has been determined, the points on the rising branch of the curve are checked, to confirm that the temperature corrections are negligible. If this is not the case, the points are discarded. Corrections have not been made for the two experimental points in figure 5 corresponding to the two highest values of $(\overline{\eta}_0 D)$, as figure 13 could not be read to sufficient accuracy in the region required for this purpose.

Appendix D. Results of calculations based on experiments by Crook (1963) and by Smith (1960)

Table 1. Results of Crook (1963); steel surfaces

figure no. in original paper	mean peripheral speed (cm/s)	load per unit width (dyn cm^{-1})	viscosity of oil at atmospheric pressure at temperature of disks (P)	max. Hertzian pressure (dyn/cm^2)	$\bar{\eta}_0$ effective viscosity at low shear rate (P)	$\log_{10}K$
22 (*a*)	400	$7{\cdot}5\times10^7$	0·4	$3{\cdot}8\times10^9$	660	1·1
22 (*b*)	400	$1{\cdot}25\times10^8$	0·4	$4{\cdot}9\times10^9$	1550	0·6
	600	$1{\cdot}25\times10^8$	0·4	$4{\cdot}9\times10^9$	1550	0·8
	400	$1{\cdot}25\times10^8$	0·7	$4{\cdot}9\times10^9$	3400	0·7
22 (*c*)	400	$2{\cdot}0\times10^8$	0·4	$6{\cdot}2\times10^9$	3500	0·6
25	400	$1{\cdot}2\times10^8$	1·4	$4{\cdot}9\times10^9$	7000	0·9

Table 2. Results of Smith (1960); estimated film thickness

Film thickness (cm) estimated from theory of Grubin (1949)

temp. of disks (°C)	oil viscosity (P)	γ*	film thickness (cm) at rolling speed U_1 (cm/s)							
			8	16	32	64	128	256	512	1024
23	0·156	$2{\cdot}9\times10^{-9}$	$4{\cdot}0\times10^{-6}$	$5{\cdot}6\times10^{-6}$	$1{\cdot}1\times10^{-5}$	$1{\cdot}8\times10^{-5}$	$3{\cdot}0\times10^{-5}$	$4{\cdot}9\times10^{-5}$	$8{\cdot}2\times10^{-5}$	$1{\cdot}3\times10^{-4}$
100	0·0205	$1{\cdot}6\times10^{-9}$	$6{\cdot}3\times10^{-7}$	$1{\cdot}1\times10^{-6}$	$1{\cdot}8\times10^{-6}$	$2{\cdot}9\times10^{-6}$	$4{\cdot}8\times10^{-6}$	$7{\cdot}9\times10^{-6}$	$1{\cdot}3\times10^{-5}$	$2{\cdot}2\times10^{-5}$
190	0·0080	$1{\cdot}1\times10^{-9}$	$2{\cdot}4\times10^{-7}$	$3{\cdot}9\times10^{-7}$	$6{\cdot}5\times10^{-7}$	$1{\cdot}1\times10^{-6}$	$1{\cdot}8\times10^{-6}$	$3{\cdot}0\times10^{-6}$	$4{\cdot}9\times10^{-6}$	$8{\cdot}1\times10^{-6}$

* Pressure coefficient of viscosity, dyn^{-1} cm^2, estimated.

Table 3. Results of Smith (1960); steel surfaces

figure no. in original reference	bulk disk temp. (°C)	rolling speed U_1 (cm/s)	max. Hertz pressure (dyn/cm^2)	effective viscosity $\bar{\eta}_0$ (P)	$\log_{10} K$
8	23	1024	$1{\cdot}06\times10^{10}$	$4{\cdot}2\times10^{4}$	0·5
	100	1024	$1{\cdot}06\times10^{10}$	$3{\cdot}0\times10^{2}$	0·9
9	23	1024	$1{\cdot}7\times10^{10}$	$1{\cdot}1\times10^{5}$	0·2
	100	1024	$1{\cdot}7\times10^{10}$	$7{\cdot}6\times10^{3}$	0·4
10	23	256	$2{\cdot}7\times10^{10}$	$2{\cdot}1\times10^{5}$	$\bar{1}{\cdot}9$
	100	256	$2{\cdot}7\times10^{10}$	$2{\cdot}3\times10^{4}$	0·0
	190	256	$2{\cdot}7\times10^{10}$	$5{\cdot}3\times10^{2}$	0·3

Table 4. Results of Smith (1960); tungsten carbide surfaces

figure no. in original reference	bulk disk temp. (°C)	rolling speed U_1 (cm/s)	Max. Hertz pressure (dyn/cm^2)	effective viscosity $\bar{\eta}_0$ (P)	$\log_{10} K$
12	23	256	$1{\cdot}65\times10^{10}$	$1{\cdot}4\times10^{5}$	0·1
	100	256	$1{\cdot}65\times10^{10}$	$6{\cdot}6\times10^{3}$	0·3
	190	256	$1{\cdot}65\times10^{10}$	$1{\cdot}1\times10^{2}$	0·6
13	23	256	$2{\cdot}64\times10^{10}$	$5{\cdot}4\times10^{5}$	$\bar{1}{\cdot}8$
	100	256	$2{\cdot}64\times10^{10}$	$7{\cdot}3\times10^{4}$	0·1
	190	256	$2{\cdot}64\times10^{10}$	$8{\cdot}4\times10^{3}$	0·4
14	23	256	$4{\cdot}2\times10^{10}$	$7{\cdot}1\times10^{5}$	$\bar{1}{\cdot}65$
	23	256	$5{\cdot}25\times10^{10}$	$9{\cdot}0\times10^{5}$	$\bar{1}{\cdot}6$
	100	256	$4{\cdot}2\times10^{10}$	$1{\cdot}3\times10^{5}$	$\bar{1}{\cdot}75$
	190	256	$4{\cdot}2\times10^{10}$	$3{\cdot}0\times10^{4}$	$\bar{1}{\cdot}95$

Table 5. Results of Smith (1960); steel on hard aluminium

Figures 5 and 6 in original reference; bulk temperature 23 °C; mean stress $4{\cdot}1\times10^{9}$ dyn/cm^2

rolling speed U_1 (cm/s)	effective viscosity $\bar{\eta}_0$ (P)	$\log_{10} K$
8	$1{\cdot}3\times10^{5}$	0·75
16	$5{\cdot}1\times10^{4}$	0·8
32	$4{\cdot}2\times10^{4}$	0·7
64	$2{\cdot}8\times10^{4}$	0·9
128	$1{\cdot}4\times10^{4}$	0·9
256	$7{\cdot}4\times10^{3}$	0·8
512	$5{\cdot}4\times10^{3}$	0·8

Appendix E. Analysis of variation of ($\bar{\eta}_0 K^2$) with temperature and pressure

From table 4 of appendix D, the results in table 6 may be calculated from the work of Smith (1960). The conditions used are tungsten carbide rollers, rolling speed 256 cm/s. An analysis of variance is given in table 7.

Table 6. Values of $\log_{10}$ ($\bar{\eta}_0 K^2$, P)

max. Hertz pressure 10^{10} dyn/cm^2	temperature (°C) 23	100	190	mean
1·65	5·3	4·4	3·2	4·3
2·64	5·3	5·1	4·7	5·0
4·2	5·2	4·9	4·4	4·8
mean	5·3	4·8	4·1	

Table 7. Analysis of variance of entries in table 6

source	sum of squares	degrees of freedom	mean square
temperatures	2·07	2	1·03
pressures	0·86	2	0·43
interaction	0·66	4	0·17
total	3·59	8	—

The effect of pressure is significant only at the 20 % confidence level when it is tested against the interaction, but the effect of temperature is significant at approximately the 5 % level.

The relation between the mean values of ($\bar{\eta}_0 K^2$) for any one temperature and the viscosity $(\eta_0)_0$ at low rates of shear at atmospheric pressure and at that temperature, is given in table 8. The linear regression of $\log_{10}(\bar{\eta}_0 K^2)$ on $\log_{10}(\eta_0)_0$ contributes 1·86 to the sum of squares attributed to the temperature effect in table 7, and this is significant at a level considerably better than 5 % when it is tested against the interaction.

Table 8. Variation of $[\bar{\eta}_0 K^2/(\eta_0)_0]$

(All viscosities in poise.)

temperature (°C)	mean $\log_{10}(\bar{\eta}_0 K^2)$	$\log_{10}\{(\eta_0)_0\}$	$\log_{10}[\bar{\eta}_0 K^2/(\eta_0)_0]$
23	5·3	$\bar{1}$·2	6·1
100	4·8	$\bar{2}$·3	6·5
190	4·1	$\bar{3}$·9	6·2

The effect of different roller materials may be seen from a comparison of the entries in table 3, for steel surfaces, with those of table 4, for tungsten carbides surfaces, for a rolling speed of 256 cm/s, and approximately equal maximum Hertz pressures of $2·69 \times 10^{10}$ and $2·64 \times 10^{10}$ dyn/cm^2 respectively. The results are shown in table 9.

Table 9. Effect of roller materials on K

$\log_{10} K$

material	temperature (°C) 23	100	190
steel	$\bar{1}$·9	0·0	0·3
tungsten carbide	$\bar{1}$·8	0·1	0·4

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